APPLIED MATHEMATICS 1

(CBCGS 2016)

Q1]a) If cosαcosβ = $\frac{x}{2}$, sinαsinβ$\frac{y}{2}$, prove that :- **(3)**

$$\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{4x}{x^2-y^2}$$

Solution:- cosαcosβ = $\frac{x}{2}$ and sinαsinβ= $\frac{y}{2}$ ....(given)

$$\sec(\alpha - i\beta) = \frac{1}{\cos(\alpha - i\beta)} = \frac{\frac{1}{\cos\alpha\cos i\beta+\sin\alpha\sin i\beta}}{\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta}\ \frac{1}{\frac{x}{2}+\frac{iy}{2}} = \frac{2}{x+iy} \quad \text{......................(1)}$$

similarly for sec(α + iβ) we get ,

$$\sec(\alpha + i\beta) = \frac{2}{x-iy} \quad \text{..................(2)}$$

from (1) and (2)

$$\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \frac{2}{x+iy} + \frac{2}{x-iy} = \frac{4x}{x^2-y^2}$$

Q1]b) If Z = log(e^x + e^y) show that rt - s^2 = 0 where r = $\frac{\partial^2 Z}{\partial x^2}$, t = $\frac{\partial^2 Z}{\partial y^2}$ s = $\frac{\partial^2 Z}{\partial x\,\partial y}$

Solution :- **(3)**

Z = log(e^x + e^y)

(1) $\frac{\partial z}{\partial x} = \frac{e^x}{(e^x+e^y)}$ $\qquad \frac{\partial^2 Z}{\partial x^2} = \frac{e^x(e^x+e^y)-e^x(e^x)}{(e^x+e^y)^2} = \frac{e^{2x}+e^{xy}-e^{2x}}{(e^x+e^y)^2}$

$$r = \frac{\partial^2 Z}{\partial x^2} = \frac{e^{xy}}{(e^x+e^y)^2} \quad \text{.............(1)}$$

(2) $\frac{\partial z}{\partial y} = \frac{e^y}{(e^x + e^y)}$ $\frac{\partial^2 Z}{\partial y^2} = \frac{e^y(e^x + e^y) - e^y(e^y)}{(e^x + e^y)^2} = \frac{e^{2y} + e^{xy} - e^{2y}}{(e^x + e^y)^2}$

$t = \frac{\partial^2 Z}{\partial y^2} = \frac{e^{xy}}{(e^x + e^y)^2}$ ....(2)

(3) $\frac{\partial z}{\partial x} = \frac{e^x}{(e^x + e^y)}$ $s = \frac{\partial^2 Z}{\partial x\, \partial y} = \frac{e^{xy}}{(e^x + e^y)^2}$ ....(3)

From (1) , (2) and (3) we get ,

$rt = \left(\frac{e^{xy}}{(e^x + e^y)^2}\right) \times \left(\frac{e^{xy}}{(e^x + e^y)^2}\right) = \left(\frac{e^{xy}}{(e^x + e^y)^2}\right)^2 = \left(\frac{e^{2xy}}{(e^x + e^y)^2}\right)$ ....(4)

$s^2 = \left(\frac{e^{xy}}{(e^x + e^y)^2}\right)^2 = \left(\frac{e^{2xy}}{(e^x + e^y)^2}\right)$ ....(5)

From (4) and (5) we get,

$rt - s^2 = \left(\frac{e^{2xy}}{(e^x + e^y)^2}\right) - \left(\frac{e^{2xy}}{(e^x + e^y)^2}\right) = 0.$

Hence proved $rt - s^2 = 0$

Q1] c) If x=uv , y = $\frac{u+v}{u-v}$. find $\frac{\partial(u,v)}{\partial(x,y)}$. **(3)**

Solution:- $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$x = uv, \quad y = \frac{u+v}{u-v}$ ....(given)

we know that JJ' = 1 ....(1)

the equation can also be solved by this following method.

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\frac{\partial x}{\partial u} = \partial(uv) = v.$ .... (2)

$\frac{\partial x}{\partial v} = \partial(uv) = u.$ ....(3)

$$\frac{\partial y}{\partial u} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)-(u+v)}{(u-v)^2} \quad u-v-u+v/u-v^2 = \frac{-2v}{(u-v)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

$$\frac{\partial y}{\partial v} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)+(u+v)}{(u-v)^2} = \frac{2u}{(u-v)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

From equation (2), (3), (4), (5) we get,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix} = \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}.$$

From (1) we get,

JJ’ = 1

$J \times \frac{4uv}{(u-v)^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots$(let $J' = \frac{4uv}{(u-v)^2}$)

Hence $J = \frac{(u-v)^2}{4uv}$.

$\therefore e^{2\varphi} = \cot\frac{\alpha}{2}$

Q1] d) If y = $2^x \sin^2$**xcosxfind** y_n **(3)**

Solution :- $2^x = e^{x\log 2} = e^{ax}$ where a = log2

$$\frac{2\sin^2 x\cos x}{2} = \frac{\sin^1 x\cos x.\sin x \times 2}{2} = \frac{\sin x.\sin 2x}{2} = \frac{2\sin x.\sin 2x}{2\times 2} = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$\therefore \sin^2 x\cos x = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$Y = \frac{e^{ax}\cos x}{4} - \frac{e^{ax}\cos 3x}{4}$$

$$y_n = \tfrac{1}{4} r_1^n e^{ax} \cos\left(x+n\varphi_1\right) - \tfrac{1}{4} r_2^n e^{ax} \cos\left(3x+n\varphi_2\right)$$

$$y_n = \tfrac{1}{4} r_1^n 2^{1x} \cos\left(x+n\varphi_1\right) - \tfrac{1}{4} r_2^n 2^{1x} \cos\left(3x+n\varphi_2\right)$$

$$r_1 = \sqrt{(\log 2)^2} + 1 \qquad r_2 = \sqrt{(\log 2)^2} + 3^2$$

$\varphi_1 = \tan^{-1}\left[\frac{1}{\log 2}\right] \qquad \varphi_2 = \tan^{-1}\left[\frac{3}{\log 2}\right]$

Q1]e) Express the matrix as the sum of symmetric and skew symmetric matrices.

Solution:- **(4)**

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A+A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}(A+A') = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

$P' = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

Hence P = P'. P is a symmetric matrix.

Let $Q = \frac{1}{2}(A-A') = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$

$$Q' = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Hence Q = Q'. Q is a skew symmetric matrix.

Q1] f) Evaluate $\lim_{x\to 0} \frac{e^{2x} - (1+x)^2}{x\log(1+x)}$ **(4)**

Solution :- $\lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{\frac{x\log(1+x)}{x}.x} = \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x.x}$

$= \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x^2}$

Applying L-Hospital rule

$\lim_{x\to 0} 1. \frac{2e^{2x} - 2(1+x)^1}{2x} = \lim_{x\to 0} 1. \frac{4e^{2x} - 2}{2} = \frac{4e^0 - 2}{2} = 1.$

Q2]a) Show that the roots of x^5=1 can be written as 1,α^1,α^2,α^3,α^4 . hence show that (1 -α^1)(1-α^2)(1-α^3)(1-α^4) = 5. (6)

Solution:- $x^5 = 1 = \cos 0 + i\sin 0$

$\therefore x^5 = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore x^1 = (\cos(2k\pi) + i\sin(2k\pi))^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting k =0,1,2,3,4 we get the five roots as

$x_0 = \cos 0 + i\sin 0 = 1$, $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$,

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$, $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$ we see that $x_2 = \alpha^2$, $x_3 = \alpha^3$, $x_4 = \alpha^4$

$\therefore$ the roots are 1, α, α^2, α^3, α^4 and hence

$\therefore\ x^5 - 1 = (x-1)(x-\alpha\alpha(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore\ \frac{x^5-1}{(x-1)} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore\ (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x^1 + 1.$

Putting x=1, we get

$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$

Q2]b) Reduce the following matrix to its normal form and hence find its rank.

Solution:- **(6)**

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$2R_3 - R_1$ $\quad R_1 - R_3$ $\quad R_1 - R_2$ $\quad R_2 - R_4$

$$\begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_2 - R_4$ $\quad 4R_2 - R_3$ $\quad R_1 + 2R_4$ $\quad R_3/(-9)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$

$2R_3 + R_4$ $\quad 6C_2 - C_4$ $\quad 29/9C_3 + C_4$ $\quad R_4\ /(-13/9)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence the given matrix is converted to its normal form

Q2] c) Solve the following equation by Gauss-Seidel method up to four iterations

4x-2y-z = 40, x-6y+2y = -28, x-2y+12z = -86. **(8)**

Solution:- we first write the equation as

x = ¼ $[40+2y+z]$ ....(1)

y = $\frac{1}{6}[28+x+2z]$ ....(2)

z = $\frac{1}{12}[-86-x+2y]$ ....(3)

(i) FIRST ITERATION :-

we start with the approximation y=0 , z=0 and then we get from (1),

$\therefore x_1 = ¼\,(40) = 10$

We use this approximation to find y i.e. put x=0, z=0 in (2)

$\therefore y_1 = \frac{1}{6}[28+10+2(0)] = 6.3333$

We use these values of x_1 and y_1 to find z_1 i.e. we put x =4 , y = 6.3333 in (3),

$\therefore z_1 = \frac{1}{12}[-86-10+2(6.3333)] = -6.944$

(ii) SECOND ITERATION :-

We use latest values of y and z to find x i.e. we put y =6.3333 , z = -6.9444 in (1)

$\therefore x_2 = ¼\,[40+2(6.3333)-6.9444] = 11.4306$

We use this approximation to find y i.e. put x=11.4306, z=-6.9444 in (2)

$\therefore y_2 = \frac{1}{6}[28+11.4306+2(-6.9444)] = 4.2569$

i.e. we put x =11.4306 , y = 4.2569 in (3),

$\therefore z_1 = \frac{1}{12}[-86-11.4306+2(4.2569)] = -7.4097$

(iii) THIRD ITERATION :-

We use latest values of y and z to find x i.e. we put y =4.2569 , z = -7.40974 in (1)

$\therefore x_2 = ¼\,[40+2(4.2569)-7.4097] = 10.2760$

We use this approximation to find y i.e. put x=10.2760 , z = -7.4097 in (2)

$\therefore y_2 = \frac{1}{6}[28+10.2760+2(-7.4097)] = 3.9094$

i.e. we put x =10.2760 , y = 3.9094 in (3),

$\therefore z_1 = \frac{1}{12}[-86-10.2760+2(3.9094)] = -7.3714.$

(iv) FOURTH ITERATION:-

We use latest values of y and z to find x i.e. we put y =3.9094 , z = -7.3714 in (1)

$\therefore x_2 = \frac{1}{4}[40+2(3.9094)-7.3714] = 10.1118$

We use this approximation to find y i.e. put x=10.1118 , z = -7.3714 in (2)

$\therefore y_2 = \frac{1}{6}[28+10.1118+2(-7.3714)] = 3.8948$

i.e. we put x =10.1118 , y = 3.8448 in (3),

$\therefore z_1 = \frac{1}{12}[-86-10.1118+2(3.8948)] = -7.3602.$

Hence , upto two places of decimals

x = 10.11 , y = 3.89, z = -7.36.

Q3] a) Investigate for what values of μ and λ the equations x+ y+ z = 6, x + 2y + 3z = 10, x + 2y + λz = μ has

1) **No solution**
2) **A unique solution**
3) **Infinite number of solutions.** **(6)**

Solution:- we have $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

By $R_2 - R_1$, $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

i) The system has unique solution if the coefficient matrix is non-singular (or the rank A, r= the number of unknowns , n =3).

This requires $\lambda - 3$ not equal to 0,

Hence λ is not equal to 3.

Hence the system has unique solution.

ii) If $\lambda = 3$ the coefficient matrix and the augmented matrix becomes

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \text{ and } \begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 0 & 0 & \mu-10\end{bmatrix}$$

The rank of A = 2 the rank of [A,B] will be also 2 if $\mu = 10$.

Thus if $\lambda = 33$and $\mu = 10$ the system is consistent. But the rank of A (= 2) is less than the number of unknowns (=3). Hence the equation will posses infinite solutions.

iii) If $\lambda = 3$ and $\mu \neq 10$, the rank of A=2, and the rank of [A,B] = 3. They are not equal and the equations will be inconsistent and will not posses any solution.

Q3]b) If u = $x^2 + y^2 + z^2$ where x = e^t , y = e^tsint , z = e^tcost

Prove that $\frac{du}{dt} = 4e^{2t}$. **(6)**

Solution:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$\frac{\partial u}{\partial x} = 2x \ , \ \frac{\partial u}{\partial y} = 2y \ , \ \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = e^t \ , \ \frac{dy}{dt} = e^t(\sin t+\cos t) \ , \ \frac{dz}{dt} = e^t(-\sin t+\cos t)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$= 2x(e^t) + 2y(e^t(\sin t+\cos t)) + 2z(e^t(-\sin t+\cos t))$

$= 2x(x) + 2y(y+z) + 2z(z-y)$

$= 2x^2 + 2y^2 + 2z^2 + 2xy - 2xy$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2 + y^2 + z^2)$

$= 2u$ ....(1)

$u = x^2 + y^2 + z^2 = ((e)^t)^2 + (e^t\sin t)^2 + (e^t\cos t)^2$

$= e^{2t} + e^{2t}(\sin^2 t+\cos^2 t)$

$= e^{2t} + e^{2t} = 2e^{2t}$

Substituting value of u in equation (1)

$\frac{du}{dt} = 2u = 2(2e^{2t}) = 4e^{2t}$

Hence proved

$\frac{Du}{dt} = 4e^{2t}.$

Q3]c) i) Show that $\sin(e^x-1) = x^1 + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots\ldots\ldots$ **(4)**

Solution :- We have $\sin(e^x-1) = \sin\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots\ldots\ldots\ldots\ldots\ldots -1\right)$

$\therefore \sin(e^x-1) = \sin\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots\ldots\ldots\ldots\right)$

But $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots$

$\therefore \sin(e^x-1) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots - \frac{1}{6}\left(x+\frac{x^2}{2}+\ldots\ldots\right)^3 + \ldots\ldots$

$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots - \frac{x^3}{6} - \frac{x^4}{4} + \ldots\ldots$

$= x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots\ldots$

Q3]c) ii) Expand $2x^3 + 7x^2 + x - 6$ **in powers of (x-2)** **(4)**

Solution :- Let $f(x) = 2x^3 + 7x^2 + x - 6$ and $a=2$

$\therefore f'(x) = 6x^2 + 14x + 1$, $f''(x) = 12x + 14$, $f'''(x) = 12$

$\therefore f(2) = 45$, $f'(2) = 53$, $f''(2) = 38$, $f'''(2) = 12$.

Now, $f(x) = f(a) + f(x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\therefore$ f(x) = f(2) + (x-2)f'(2) + $\frac{(x-2)^2}{2!}$f''(2) +..................

$2x^3 + 7x^2 + x - 6 = 45 + (x-2).53 + (x-2)^{2.19} + (x-2)^{3.2}$

Q4] a) If x = u+v+w, y = uv+vw+uw, z = uvw and φ is a function of x,y and z.

Prove that $x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$ **(6)**

Solution:- φ is a function of x,y and z and x,y,z are themselves functions of u,v,w.

$$\therefore \frac{\partial\varphi}{\partial u} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial u} = \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+w) + \frac{\partial\varphi}{\partial z}vw$$

And $$\frac{\partial\varphi}{\partial v} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial v}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(u+w) + \frac{\partial\varphi}{\partial z}.uw$$

And $$\frac{\partial\varphi}{\partial w} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial w} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial w}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+u) + \frac{\partial\varphi}{\partial z}.uv$$

Multiplying (1) by u , (2) by v, (3) by w and add

$$\therefore u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w} = (u+v+w)\frac{\partial\varphi}{\partial x} + [(uv+uw) + (vu+vw)+(wv + wu)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= (u+v+w)\frac{\partial\varphi}{\partial x} + [2(uv+vw+uw)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z}$$

$$\therefore x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$$

Q4] b) If tan(θ + iφ) = tanα + isecα

Prove that 1) $e^{2\varphi}$ **=** $\cot\frac{\alpha}{2}$ **2)** $2\theta = n\pi + \frac{\pi}{2} + \alpha$. **(6)**

Solution :- tan(θ + iφ) = tanα + isecα $\therefore$ tan(θ - iφ) = tanα - isecα

$\therefore\ \tan 2\theta = \tan[(\theta + i\varphi i\varphi + (\theta - i\varphi)] \quad = \quad \dfrac{\tan(\theta+i\varphi) + \tan(\theta-i\varphi)}{1-\tan(\theta+i\varphi)\tan(\theta-i\varphi)}$

$$= \frac{\tan(\theta+i\sec\alpha) + \tan(\theta-i\sec\alpha)}{1-\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)} \quad = \quad \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)} = \frac{2\tan\alpha}{-2\tan^2\alpha}\text{-}\cot\alpha \quad = \tan\left(\frac{\pi}{2} + \alpha\right)$$

$\therefore\ 2\theta = n\pi + \dfrac{\pi}{2} + \alpha.$ (general value).

Again $\tan(2i\varphi) = \tan[(\theta + i\varphi) - (\theta - i\varphi)]$

$$= \frac{\tan(\theta+i\sec\alpha) - \tan(\theta-i\sec\alpha)}{1+\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)}$$

$\therefore\ i\tanh 2\varphi = \dfrac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha \qquad \therefore\ \tanh 2\varphi = \cos\alpha$

$\therefore\ 2\varphi = \tanh\text{^}(-1)\ (\cos\alpha)\ (\cos\alpha \dfrac{1}{2}\log\left[\dfrac{1+\cos\alpha}{1-\cos\alpha}\right] \qquad = \dfrac{1}{2}\log\left[\dfrac{2\cos^2\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)}\right]\text{og}\cot\dfrac{\alpha}{2}$

Q 4]c) Find the roots of the equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 2.1 correct to 3 places of decimals using Regula Falsi method.

Solution:- **(8)**

Given that a=2 and b =2.1.

$f(2) = (2)^4 + (2)^3 - 7(2)^2 - 2 + 5 = -1.$

$f(2.1) = (2.1)^4 + (2.1)^3 - 7(2.1)^2 - (2.1) + 5 = 0.739100.$

$x_1 = \dfrac{af(b)-bf(a)}{f(b)-f(a)} = (2\times0.73910—1))\times(2.1)0.739100—10.739100—1.05750.$ ……………(1)

$f((x_1) = (2.05750)^4 + (2.05750)^3 - 7(2.05750)^2 - 2.05750 + 5$

$= -0.05973.$

$x_2 = \dfrac{af(x_1)- x_1f(a)}{f(x_1)-f(a)} = (2\times(-0.05973)—1))\times(2.05750)0.05973—1.05750)0.05973—12$

……………(2)

$f((x_2) = (2.061152)^4 + (2.061152)^3 - 7(2.061152)^2 - 2.061152 + 5$

$= 0.005326.$

$x_3 = \dfrac{af(x_2)- x_2f(a)}{f(x_2)-f(a)} = (2\times(0.005326)—1))\times(2.061152)0.005326—1.061152)0.005326—1.$

………………(3)

$f((x_3) = (2.06082)^4 + (2.06082)^3 - 7(2.06082)^2 - 2.06082 + 5$

$= -0.000582.$

$x_4 = \frac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)} = (2\times(-0.000582)—1))\times(2.06082)0.000582—12.06082)0.000582—1$

………………(4)

Hence from (4) and (3) iteration we get that value of x is coinciding.

Therefore the final value of x is 2.0608.

Q5] a) If $y = (x + \sqrt{x^2 - 1})^m$, Prove that

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$ (6)

Solution:-

$y = (x + \sqrt{x^2 - 1})^m$

taking + sign before the radical

$\therefore y_1 = m[(x + \sqrt{x^2 - 1})^{m-1}].[1+\frac{x}{\sqrt{x^2-1}}]$

$= m(x + \sqrt{x^2 - 1})^m). \frac{x}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

$\sqrt{x^2 - 1} . y_1 = my$

Differentiating again w.r.t x,

$\sqrt{x^2 - 1} . y_2 + \frac{x}{\sqrt{x^2-1}} y_1 = m y_1$

$(x^2 - 1) y_2 + xy_1 = m\sqrt{x^2 - 1}. y_1 = m.my = my^2$

$(x^2 - 1) y_2 + xy_1 - my^2 = 0$

Hence after applying lebnitz's theorem we get,

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Q5]b) Using the encoding matrix $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ encode and decode the message

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Solution:-

A B C D E F G H I J K L M N O P Q R S T U V W X

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

* Y. Z

27. 25. 26

I * L O V E * M U M B A I *

9 27 12 15 22 5 27 13 21 13 2 1 9 27

Encoding the message includes the following process.

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}9 & 12 & 22 & 27 & 12 & 2 & 9\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}9+27 & 12+15 & 22+5 & 27+13 & 12+13 & 2+1 & 9+27\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}36 & 27 & 27 & 40 & 25 & 3 & 36\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

Hence the encoded message we get as ,

36, 27, 27, 15, 27, 5, 40, 13, 25, 13, 3, 1, 36, 27.

Now the process of decoding is as follows.

Inverse of $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}36 & 27 & 27 & 40 & 25 & 3 & 36\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}36-27 & 27-15 & 27-5 & 40-13 & 25-13 & 3-1 & 36-27\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

$$= \begin{bmatrix}9 & 12 & 22 & 27 & 12 & 2 & 9\\27 & 15 & 5 & 13 & 13 & 1 & 27\end{bmatrix}$$

Hence the original message is obtained again after encoding and decoding.

Q5]c) i) Considering only principal values separate into real and imaginary parts

$i^{\log(i+1)}$. **(4)**

Solution:- let $Z = i^{\log(i+1)}$ $\therefore$ logZ = log(1+i).logi

But $\log(i+1) = \log\sqrt{2} + i\tan^{-1} 1 = \log\sqrt{2} + i\frac{\pi}{4}$ and $\log i = i.\frac{\pi}{2}$

$\therefore \log Z = (\log\sqrt{2} + i\frac{\pi}{4}).\ i.\frac{\pi}{2} = [\frac{1}{2}\log 2 + i\frac{\pi}{4}\pi)/2 = \frac{-\pi^2}{8} + i\frac{\pi}{4}\log 2 = e^{-\frac{\pi^2}{8}+i\theta} = e^{-\frac{\pi^2}{8}.i\theta}$

where $\theta = \frac{\pi}{4}\log 2 = e^{-\frac{\pi^2}{8}}[\cos\theta + i\sin\theta i\sin\theta$

$\therefore$ Real part of $Z = e^{-\frac{\pi^2}{8}}\cos\theta = e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$ ∴nary part of $Z = e^{-\frac{\pi^2}{8}}\sin\left(\frac{\pi}{4}\log 2\right)$

Q5]c) ii) Show that $i\log(\frac{x-i}{x+i}) = \pi - 2\tan^{-1} x$ **(4)**

Solution:- we have $\log(x+i) = ½ \log(x^2+1) + i\tan^{-1}\frac{1}{x}$

and $\log(x-i) = ½ \log(x^2+1) - i\tan^{-1}\frac{1}{x}$

$\log(\frac{x-i}{x+i}) = \log(x-i) - \log(x+i)$

$= -2i\tan^{-1}\frac{1}{x} = -2i(\frac{\pi}{2} - \tan^{-1} x)$

$\therefore \log(\frac{x-i}{x+i}) = -i(\pi - 2\tan^{-1} x)$

$\therefore i\log(\frac{x-i}{x+i}) = (\pi - 2\tan^{-1} x)$.

Q6]a) Using De Moivre's theorem prove that

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$ **(6)**

Solution:- Let as above $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$(2\cos\theta)^6 = (x + \frac{1}{x})^6$

$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x^1.\frac{1}{x^5} + \frac{1}{x^6}$

$= x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$ ....(1)

$(2i\sin\theta)^6 = (x - \frac{1}{x})^6$

$= x^6 - 6x^5 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$ ....(2)

$(2\sin\theta)^6 = x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}$

Subtracting (2) from (1),

$2^6(\cos^6\theta - \sin^6\theta) = [x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}] -$

$[- x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}]$

$= 2(x^6 + \frac{1}{x^6}) + 15(x^2 + \frac{1}{x^2})$

$= 2\cos6\theta + 15\cos2\theta$ ....$[(x^6 + \frac{1}{x^6}) = \cos6\theta]$

$\therefore 2^6 (\cos^6\theta - \sin^6\theta) = 2\cos6\theta + 15\cos2\theta.$

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos6\theta+15\cos2\theta)$

Q6] b) If $u = \sin^{-1}(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}})^{1/2}$, Prove that

$x^2\frac{\partial^2 u}{\partial x^2}$ **+ 2xy**$\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tanu}{144}(\tan^2 u + 13)$ **(6)**

Solution:-

$Z = \sin u = \sqrt{(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}})} = f(u) = F(X,Y)$ say .

Putting X = xt, Y = yt

$F(X,Y) = \sqrt{\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = \sqrt{\left(\frac{(xt)^{\frac{1}{3}} + (yt)^{\frac{1}{3}}}{(xt)^{\frac{1}{2}} - (yt)^{\frac{1}{2}}}\right)} = \sqrt{\frac{t^{1/3}}{t^{1/2}}\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = t^{-1/12}f(x,y)$

Thus Z = f(u) = sinu is a homogenous function of x, y of degrees 1/12

Hence , by the above corollary.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

Where , $g(u) = n\frac{f(u)}{f'(u)} = \frac{-1}{12}.\frac{\sin u}{\cos u} = \frac{-1}{12}\tan u$

$g'(u)-1 = \frac{-1}{12}\sec^2 u - 1 = \frac{-1}{12}(1-\tan^2 u) - 1 = \frac{-1}{12}\tan^2 u - \frac{13}{12}$

$= \frac{-1}{12}(\tan^2 u + 13)$

$\therefore$ $g(u)[g'(u)-1] = \left(\frac{-1}{12}\tan u\right)\left(\frac{-1}{12}(\tan^2 u + 13)\right)$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13)$$

Q6]c) Find the maxima and minima of $x^3y^2(1-x-y)$ **(8)**

Solution :- we have $f(x) = x^3y^2(1-x-y)$

Step 1 :- $f_x = y^2[3x^2(1-x-y) - x^3] = y^2(3x^2-4x^3-3x^2y)$

$= (3x^2y^2-4x^3y^2-3x^2y^3)$

f_y = x^3 [2y(1 - x - y)y(1-x-y)y^2] = $x^3(2y-2xy-3y^2)$

$= (2yx^3-2x^4y-3x^3y^2)$

$\therefore$ $f_{xx} = 6y^2x^1 - 12x^2y^2 - 6x^1y^3$

$\therefore$ $f_{xy} = 6y^1x^2 - 8x^3y^1 - 9x^2y^2$

$\therefore$ $f_{yy} = 2x^3 - 2x^4 - 6x^3$

Step 2:- we now solve for $f_y = 0$, $f_x = 0$

$\therefore 3y^2x^2 - 4x^3y^2 - 3x^2y^3 = 0$ i.e. $y^2x^2(3-4x-3y) = 0$

And $2y^1x^3 - 2x^4y^1 - 3x^3y^2 = 0$ i.e. $y^1x^3(2-2x-3y) = 0$

$\therefore$ x= 0 , y = 0 and $(3\text{-}4x\text{-}3y) = 0$, $2 - 2x - 3y3y\ 0$

Subtracting we get 1-2x = 0

$\therefore$ x = ½ $\therefore$ 3y = 3-4(1/2) = 1 $\therefore$ y = $\frac{1}{3}$

$\therefore$ (0,0) and (1/2 , 1/3) are stationary points.

Step 3:- at x = 0, y = 0 , r = 0 , s = 0 , t = 0 $\therefore$ rt - s^2 = 0

At x = ½ , y =1/3

r = f_{xx} = $6(1/2)(1/9) - 12(1/4)(1/9) - 6(1/2)(1/27)/9)\text{-}12(1/4)(1/9)\text{-}6(1/2)(1/27)\frac{1}{3} - \frac{1}{3} - \frac{1}{9}$

= $-\frac{1}{9}$

s = f_{xy} = $6(1/4)(1/3) - 8(1/8)(1/3) - 9(1/4)(1/9)3)\text{-}8(1/8)(1/3)\text{-}9(1/4)(1/9)$ = ½ - $\frac{1}{3}$ - ¼

= $-\frac{1}{12}$

t = f_{xy} = $2(1/8) - 2(1/16) - 6(1/8)(1/3)(1/16)\text{-}6(1/8)(1/3)$ = ¼ - $\frac{1}{8}$ - ¼ = $-\frac{1}{8}$

$\therefore$ rt - s^2 = $(-1/9)(-1/8) - (1/12)(1/12)9)(-1/8)\text{-}(1/12)(1/12)$ = $\frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$

And r = -1/9 <0 $\therefore$ f(x,y) is a maxima

Maximum value = $\frac{1}{8}.\frac{1}{9}\left(1\text{-}\frac{1}{2}\text{-}\frac{1}{3}\right) = \frac{1}{432}$

MUMBAI UNIVERSITY

SEMESTER 1

APPLIED MATHEMATICS SOLVED PAPER – MAY 2017

Q.1(a) Prove that $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$ **[3]**

Ans : L.H.S = $\tanh^{-1}(\sin\theta)$

We know that , $\tanh^{-1}(x) = \frac{1}{2}\log(\frac{1+x}{1-x})$

$\therefore$ L.H.S = $\frac{1}{2}\log(\frac{1+\sin\theta}{1-\sin\theta})$

R.H.S = $\cosh^{-1}(\sec\theta)$

We know that , $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$

$\therefore$ R.H.S = $\log(\sec\theta + \sqrt{\sec^2\theta - 1})$

$= \log(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta})$ ......$\{\sqrt{\sec^2\theta - 1} = \tan\theta = \frac{\sin\theta}{\cos\theta}\}$

$= \log(\frac{1+\sin\theta}{\cos\theta})$

$= \log(\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}})$

$= \log(\frac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}})$

$= \frac{1}{2}\log(\frac{1+\sin\theta}{1-\sin\theta})$

$$\therefore \tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$$

Hence Proved .

(b) Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. [3]

Ans : Let A = $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

The matrix is unitary when A.$A^{\theta} = I$.

$\therefore \quad A^{\theta} = (\bar{A})^t = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}^t = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\therefore \quad A.A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{\therefore \quad A.A^{\theta} = I}$

The given matrix is unitary is proved.

(c) If x=uv & y=$\frac{u}{v}$ prove that $JJ^1 = 1$ [3]

Ans : x =uv and y=$\frac{u}{v}$

$\therefore$ x and y are function of u and v .

$\therefore$ u =$\sqrt{xy}$ $\therefore$ v=$\sqrt{\frac{x}{y}}$ ……. { from given eqns }

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v} \quad ......(1)$$

$$J^1 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix} = \frac{-\sqrt{\frac{x}{y}}}{2\sqrt{xy}} = \frac{-v}{2u} \quad ....(2)$$

$\therefore JJ^1 = \frac{-2u}{v} \times \frac{-v}{2u} = 1$

$\boxed{\therefore JJ^1 = 1}$

Hence Proved.

(d) If z =$\tan^{-1}(\frac{x}{y})$, where x=2t , y=1-t^2, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$. [3]

Ans : $z = \tan^{-1}(\frac{x}{y})$ $x=2t$ and $y=1-t^2$

∴ z is the function of x and y & x and y are the functions of t.

$z \longrightarrow f(x,y) \longrightarrow f(t)$

$\therefore z = \tan^{-1}(\frac{2t}{1-t^2})$

Direct differentiate w.r.t t ,

$$\frac{dz}{dt} = \frac{1}{1+(\frac{2t}{1-t^2})^2} \times \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$= \frac{2(1-t^2)^2}{(1-t^2)^2+4t^2} \times [t.\frac{1}{(1-t^2)^2}(-2t) + \frac{1}{1-t^2} \times 1]$$

$$= \frac{2(1-t^2)^2}{1+t^2} \times \frac{1}{(1-t^2)^2}$$

$$\therefore \frac{dz}{dt} = \frac{2}{1+t^2}$$

Hence Proved.

(e) Find the nth derivative of cos 5x.cos 3x.cos x. [4]

Ans : let $y = \cos 5x.\cos 3x.\cos x$

$$= \frac{\cos(5x-3x)+\cos(5x+3x)}{2} . \cos x$$

$$= \frac{1}{2}[\cos 2x.\cos x + \cos 8x.\cos x]$$

$$y = \frac{1}{4}[\cos 3x + \cos x + \cos 9x + \cos 7x]$$

Take n th derivative,

$$n\ th\ derivative\ of\ \cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$y_n = \frac{1}{4}[9\cos\left(\frac{n\pi}{2}+3x\right) + \cos\left(\frac{n\pi}{2}+x\right) + 81\cos\left(\frac{n\pi}{2}+9x\right) + 49\cos\left(\frac{n\pi}{2}+7x\right)$$

(f) Evaluate : $\lim_{x \to 0}(x)^{\frac{1}{1-x}}$ [4]

Ans : Let $L = \lim_{x\to 0}(x)^{\frac{1}{1-x}}$

Take log on both the sides,

$$\therefore \log L = \lim_{x\to 0}\frac{\log x}{1-x}$$

Apply L'Hospital rule ,

$$\therefore \log L = \lim_{x\to 0}\frac{1}{x}$$

$$= 0$$

$$\boxed{\therefore\ L = e^0 = 1}$$

Q.2(a) Find all values of $(1+i)^{1/3}$ & show that their continued Product is (1+i). [6]

Ans : let $x = (1+i)^{1/3}$

$$\therefore\ x^3 = 1 + i = \sqrt{2}(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$$

$$\therefore\ x^3 = \sqrt{2}[cos\ \frac{\pi}{4} + isin\ \frac{\pi}{4}]$$

Add period $2k\pi$,

$$x^3 = \sqrt{2}[cos(\frac{\pi}{4} + 2k\pi) + isin\ (\frac{\pi}{4} + 2k\pi)]$$

By applying De Moivres theorem,

$$x = 2\sqrt{2}[cos\frac{1}{3}(\frac{\pi}{4} + 2k\pi) + isin\frac{1}{3}(\frac{\pi}{4} + 2k\pi)]$$

where k =0,1,2.

Roots are :

Put k=0 $\quad x_0 = 2\sqrt{2}e^{i\frac{\pi}{12}}$

Put k=1 $\quad x_1 = 2\sqrt{2}e^{i\frac{9\pi}{12}}$

Put k=2 $\quad x_2 = 2\sqrt{2}e^{i\frac{17\pi}{12}}$

The continued product of roots is given by ,

$$x_0 x_1 x_2 = 2\sqrt{2}e^{i\frac{\pi}{12}} \times 2\sqrt{2}e^{i\frac{9\pi}{12}} \times 2\sqrt{2}e^{i\frac{17\pi}{12}}$$

$$= 16\ \sqrt{2}\ e^{i\frac{27\pi}{12}}$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 1+i$$

The continued product of roots is (1+i).

(b) Find non singular matrices P & Q such that PAQ is in normal form

Where $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ [6]

Ans : Matrix in PAQ form is given by ,

A = P A Q

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_3,$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1, R_3 - 2R_1,$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1, C_3 + C_1,$

$$\begin{bmatrix}1&0&0\\0&-7&5\\0&-6&5\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-2&1\\0&1&0\\0&0&1\end{bmatrix}$$

$\frac{C_3}{5}$,

$$\begin{bmatrix}1&0&0\\0&-7&1\\0&-6&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-2&1/5\\0&1&0\\0&0&1/5\end{bmatrix}$$

C_2+6C_3,

$$\begin{bmatrix}1&0&0\\0&-1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-4/5&1/5\\0&1&0\\0&6/5&1/5\end{bmatrix}$$

C_3+C_2,

$$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-4/5&-3/5\\0&1&1\\0&6/5&7/5\end{bmatrix}$$

- R_1,

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&-1&3\\1&0&-2\end{bmatrix}A\begin{bmatrix}1&-4/5&-3/5\\0&1&1\\0&6/5&7/5\end{bmatrix}$$

Now A is in normal form with rank 3.

Compare with PAQ form ,

$$P=\begin{bmatrix}0&0&1\\0&-1&3\\1&0&-2\end{bmatrix}\qquad Q=\begin{bmatrix}1&-4/5&-3/5\\0&1&1\\0&6/5&7/5\end{bmatrix}$$

(c) Find the maximum and minimum values of

$$f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$$ [8]

Ans : given : $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$ $\quad f_{xx} = 6x - 30$

$f_y = 6xy - 30y$ $\quad f_{yy} = 6x - 30$

$f_{xy} = 6y$

To find stationary values :

$f_x = 3x^2 + 3y^2 - 30x + 72 = 0$ & $f_y = 6xy - 30y = 0$

y=0 or x=5

for y=0 , x=6,4

∴ (x,y)=(6,0) , (4,0).

For x=5 , y=1,-1

∴ (x,y)=(5,1) , (5,-1)

Stationary points are : (6,0),(4,0),(5,1),(5,-1)

(i) For point (6,0) ,

r = f_{xx}=36-30=6 , s =$f_{xy} = 0$, $t = f_{yy} = 6$

rt - $s^2 = 36 > 0$ and r =6>0

function is minimum at (6,0).

$f_{min} = 108$

(ii) For point (4,0) ,

r = f_{xx}=-6 , s =$f_{xy} = 0$, $t = f_{yy} = -6$

rt - $s^2 = 36 > 0$ and r =-6<0

function is maximum at (4,0).

$f_{max} = 112$

(iii) for point (5,1) and (5,-1) ,

Thr points are neither maximum nor minimum.

∴ The maximum and minimum value of function are 112 and 108 .

Q.3(a) If u = f$(\frac{y-x}{xy}, \frac{z-x}{xz})$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$ [6]

Ans : let u = f(r,s)

$\therefore r = \frac{y-x}{xy}$ $\therefore s = \frac{z-x}{xz}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{x^2} + \frac{\partial u}{\partial s}(\frac{-1}{x^2})$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\frac{(-1)}{y^2} + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(\frac{1}{z^2})$$

$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\boxed{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$$

Hence proved.

(b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,encode & decode the message "MUMBAI". [6]

Ans : Encoding matrix : A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Message is : MUMBAI.

The given message in matrix form is :

$$B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

Encoded message in matrix form is given by ,

$$C = A.B$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$

Encoded message is : 34 21 15 2 10 9

GUOBJI

Decoded matrix is given by ,

$$B = A^{-1}.C$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

Decoded message : MUMBAI

(c) Prove that $\log[\tan(\frac{\pi}{4} + \frac{ix}{2})] = i.\tan^{-1}(sinhx)$ [8]

Ans : L.H.S $= \log[\tan(\frac{\pi}{4} + \frac{ix}{2})]$

$$= \log\left[\frac{1+tan\,(\frac{ix}{2})}{1-tan\,(\frac{ix}{2})}\right]$$

$$= \log[1 + tan\,(\tfrac{ix}{2})] - \log[1 - tan\,(\tfrac{ix}{2})]$$

$$= \log[1 + i.\tanh\tfrac{x}{2}] - \log[1 - i\tanh\tfrac{x}{2}]$$

We have ,

$$\log(a+ib) = \frac{1}{2}log(a^2+b^2) + i\tan^{-1}(\frac{b}{a})$$

$$\therefore \; = \frac{1}{2}log\left(1+tanh^2\frac{x}{2}\right) + i\tan^{-1}\left(tanh\frac{x}{2}\right) - [\frac{1}{2}log\left(1+tanh^2\frac{x}{2}\right) - i\tan^{-1}\left(tanh\frac{x}{2}\right)]$$

$$= 2i[\tan^{-1}\left(tanh\frac{x}{2}\right)]$$

$$\text{L.H.S} = 2i.\tan^{-1}\left(tanh\frac{x}{2}\right)$$

$$\text{R.H.S} = i.\tan^{-1}(sinhx)$$

We know that $sinh^{-1}x = log(x + \sqrt{1+x^2})$

$$tanh^{-1}x = \frac{1}{2}[log(\frac{x+1}{1-x})]$$

$= \quad i\tan^{-1}\left(tanh\frac{x}{2}\right)$

Also $sinh^{-1}(tanx) = tanh^{-1}(x)$

R.H.S $=$ $i.\tan^{-1}\left(tanh\frac{x}{2}\right)$

$$\log[\tan(\frac{\pi}{4}+\frac{ix}{2})]=i.\tan^{-1}(\,sinhx)$$

Q.4(a) Obtain tan 5θ in terms of tan θ & show that

$$1\text{-}10tan^2\frac{x}{10} + 5tan^4\frac{x}{10} = 0$$ [6]

Ans : we have $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Put n=5 ,

$$\therefore \cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= cos^5\theta + 5cos^4\theta.\,isin\theta + 10cos^3\theta.(isin\theta)^2$$

$$+10cos^2\theta.(isin\theta)^3 + 5cos\theta.(isin\theta)^4 + isin^5\theta$$

$$= [cos^5\theta - 10cos^3\theta.(sin\theta)^2$$

$$+ 5cos\theta.(sin\theta)^4\,] + [5cos^4\theta.\,isin\theta$$

$$- 10icos^2\theta.(sin\theta)^3 + isin^5\theta]$$

Compare real and imaginary parts

$$\cos 5\theta = [cos^5\theta - 10cos^3\theta.(sin\theta)^2 + 5cos\theta.(sin\theta)^4$$

$$\sin 5\theta = +[5cos^4\theta.sin\theta - 10cos^2\theta.(sin\theta)^3 + sin^5\theta\,]$$

$$\tan 5\theta = \frac{[5cos^4\theta.sin\theta - 10cos^2\theta.(sin\theta)^3 + sin^5\theta\,]}{[cos^5\theta - 10cos^3\theta.(sin\theta)^2 + 5cos\theta.(sin\theta)^4}$$

$$\tan 5\theta = \frac{5tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$$

put $\theta = \frac{\pi}{10}$

$$1\text{-}10\tan^2\frac{x}{10}+5\tan^4\frac{x}{10}=0$$

(b) If $y=e^{\tan^{-1}x}$. Prove that

$$(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0 \qquad [6]$$

Ans : $y=e^{\tan^{-1}x}$ ....(1)

Diff. w.r.t x ,

$$y_1=e^{\tan^{-1}x}\frac{1}{x^2+1}$$

$$(x^2+1)y_1=e^{\tan^{-1}x}=y \qquad \text{--------(from 1)}$$

Again diff. w.r.t x,

$$(x^2+1)y_2+2xy_1=y_1 \qquad \text{...............(1)}$$

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$$(uv)_n=u_nv+{}^n_1Cu_{n-1}v_1+{}^n_2Cu_{n-2}v_2+\cdots+uv_n$$

$u=(x^2+1), v=y_2$ ...for first term in eqn (1)

$u=2x\,, v=y_1$ ...for second term in eqn (1)

$$\therefore\ (1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n-y_{n+1}=0$$

$$\therefore\ (1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$$

Hence Proved.

(c) i. Express $(2x^3+3x^2-8x+7)$ in terms of (x-2) using Taylor's Series. [4]

ii. Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ [4]

Ans : i. let f(x)= $2x^3 + 3x^2 - 8x + 7$

Here a = 2

$f(x) = 2x^3 + 3x^2 - 8x + 7$ $f(2) = 19$

$f'(x) = 6x^2 + 6x - 8$ $f'(2) = 28$

$f''(x) = 12x + 6$ $f''(2) = 30$

$f'''(x) = f'''(2) = 12$

Taylor's series is :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + (x-2)28 + \frac{(x-2)^2}{2!}30 + \frac{(x-a)^3}{3!}12$$

$$\boxed{2x^3 + 3x^2 - 8x + 7 = 19 + 28(x-2) + 15(x-2)^2 + 2(x-2)^3}$$

ii. let y = $\tan^{-1} x$

diff. w.r.t x ,

$\therefore \quad y_1 = \frac{1}{x^2+1}$

Series expansion of y_1 ,

We know that ,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$\therefore \quad y_1 = 1 - x^2 + x^4 - x^5$

Integrate y_1 to find series expansion of y,

$\therefore$ y = $\int(1 - x^2 + x^4 - x^5 + \cdots)dx$

$$\therefore \; y = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

Hence Proved .

Q.5(a) If $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$ ∂

Prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$ [6]

Ans : $z = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Diff. w.r.t. x partially ,

$$\frac{\partial z}{\partial x} = x^2 \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} + \tan^{-1}\frac{y}{x} \, . 2x - y^2 \; \frac{y^2}{x^2+y^2} \times \frac{1}{y}$$

$$= \frac{x^2}{x^2+y^2} \times \frac{-y}{1} + 2x\tan^{-1}\frac{x}{y} - \frac{y^3}{x^2+y^2}$$

Diff. w.r.t y partially ,

$$\frac{\partial^2 z}{\partial y \partial x} = -x^2\left[-y.\frac{2y}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}\right] + 2\,\frac{x^2}{x^2+y^2} - \left[-y^3.\frac{2y}{(x^2+y^2)^2} + \frac{3y^2}{x^2+y^2}\right]$$

$$= \left[.\frac{2y^3x^2}{(x^2+y^2)^2} + \frac{-x^2}{x^2+y^2}\right] + 2\,\frac{x^2}{x^2+y^2} + \frac{2y^4}{(x^2+y^2)^2} - \frac{3y^2}{x^2+y^2}$$

$$= \frac{(x^2-y^2)^2 \times}{(x^2+y^2)^2}\frac{(x^2+y^2)^1}{(x^2-y^2)^1}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$$

Hence proved.

(b) Investigate for what values of μ *and* λ the equations :2x+3y+5z=9

7x+3y-2z =8

$2x+3y+\lambda z=\mu$

Have (i) no solution (ii) unique solution (iii) Infinite value [6]

Ans : **Given eqn :** 2x+3y+5z=9

7x+3y-2z =8

2x+3y+λz=μ

A X = B

$$\therefore \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

$R_3 - R_1$,

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 6 \\ 7 & 3 & -2 & 4 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

(i) **When** λ=5,$\mu \neq 9$ $then\ r(a) = 2, r(A \vdots B) = 3$

r(A)$\neq r(A \vdots B)$

No Solution.

(ii) **When** $\lambda \neq$5,$\mu \neq 9$, $r(A) = r(A \vdots B) = 3$
Unique solution exist.

(iii) **When** λ=5,$\mu = 9$ $r(A) = r(A \vdots B) = 2 < 3$
Infinite solution.

(c) Obtain the root of $x^3 - x - 1 = 0$ by Newton Raphson Method (upto three decimal places). **[8]**

Ans : **Equation :** $x^3 - 2x - 5 = 0$

$$\therefore f(x) = x^3 - 2x - 5$$

$$f(0) = -5 < 0 \ \ and \ \ f(1) = -2 < 0 \ and \ f(2) = 7 > 0.$$

Root of given eqn lies between 1 and 2.

$$f'(x) = 3x^2 + 2$$

Let take $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{7}{14} = 1.5$$

Next iteration :

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.343$$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.329$$

For next iteration :

$$\therefore \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.329 - \frac{f(1.329)}{f'(1.329)}$$

$$= 1.3283$$

The root of eqn is x = 1.3283

Q.6(a) Find tanhx if 5sinhx-coshx = 5 **[6]**

Ans : 5sinhx-coshx = 5

But sinhx = $\frac{e^x - e^{-x}}{2}$ coshx = $\frac{e^x + e^{-x}}{2}$

$$\therefore \quad 5\left[\frac{e^x - e^{-x}}{2}\right] - \left[\frac{e^x + e^{-x}}{2}\right] = 5$$

$$\therefore \quad 5e^x - 5e^{-x} - e^x - e^{-x} = 10$$

$$\therefore \quad 4e^{2x} - 10e^x - 6 = 0$$

Roots are : $e^x = 3$, $e^x = \frac{-1}{2}$

$$\therefore \text{tanhx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\left(-\frac{1}{2}\right)+2}{-5/2} = \frac{-3}{5}$$

Or

$$\therefore \text{tanhx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3-1/3}{3+1/3} = \frac{4}{5}$$

The values of tanhx are : $\frac{-3}{5}$ *or* $\frac{4}{5}$

(b) **If u = $\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *Prove that* i. $xu_x + yu_y = \frac{1}{2}\tan u$**

ii. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u.\cos 2u}{4\cos^3 u}$ **[6]**

Ans : $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put x = xt and y = yt to find degree.

$$\therefore u = \sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore \sin u = t^{1/2}.\frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{\frac{1}{2}}.f(x,y)$$

The function sin u is homogeneous with degree ½.

But sin u is the function of u and u is the function of x and y.

By Euler's theorem ,

$$xu_x + yu_y = G(u) = n.\frac{f(u)}{f'(u)} = \frac{1}{2}\tan u$$

$$\therefore xu_x + yu_y = \frac{1}{2}\tan u$$

$$\therefore x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)[G'(u) - 1]$$

$$= \frac{1}{2}\tan u\left[\frac{\sec^2 u - 2}{2}\right]$$

$= \frac{1}{4}\tan u[\frac{tan^2u-1}{1}]$

$= \frac{1}{4} \times \frac{\sin u}{\cos u}[\frac{sin^2u-cos^2u}{cos^2u}]$

$$\therefore \quad x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-sinu.cos2u}{4cos^3u}$$

Hence Proved.

(c) Solve the following system of equation by Gauss Siedal Method,

20x+y-2z=17

3x+20y-z =-18

2x-3y+20z=25 [8]

Ans : **By Gauss Seidal method ,**

Given eqn : 20x+y-2z=17

3x+20y-z =-18

2x-3y+20z=25

From given eqn : |20|>|1|+|-2|

|20|>|3|+|-1|

|20|>|2|+|-3|

The given eqn are in correct order.

$\therefore \; x = \frac{1}{20}[17 - y + 2z]$

$\therefore \; y = \frac{1}{20}[-18 - 3x + z]$

$\therefore \; z = \frac{1}{20}[25 - 2x + 3y]$

I) For 1st iteration : take $y = 0, z = 0$

$x = \frac{1}{20}[17] = 0.85$

$x = 0.85\ , z = 0$ gives $y = -1.0275$

$x = 0.85, y = -1.0275$ gives $x_3 = 1.0109$

II) For 2nd iteration : take $y = -1.0275, z = 1.0109$

$x = \frac{1}{20}[17 + 1.0275 - 2(1.0109)] = 1.0025$

$x = 1.0025, z = 1.0109$ gives $y = -0.9998$

$x = 1.0025, y = -0.9998$ gives $z = 0.9998$

III) For 3rd iteration : $y = -0.9998, z = 0.9998$

$x_1 = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 1.00$

$x = 1.00, z = 0.9998$ gives $y = -1.00$

$x = 1.00, y = -1.00$ gives $z = 1.00$

Result : $x = 1.00, y = -1.00, z = 1.00$

APPLIED MATHEMATICS-1

Dec 2017

Mumbai University Dec 2017 (CBCS) solutions.

Q.1) Answer the following **(20 marks)**

1a) Separate into real and imaginary parts of $\cos^{-1}\left(\frac{3i}{4}\right)$. **(3 marks)**

Ans. Let $a + ib = \cos^{-1}\left(\frac{3i}{4}\right)$ ....(1)

$\therefore \cos(a + ib) = \frac{3i}{4}$

$\therefore \cos(a)\cos(ib) - \sin(a)\sin(ib) = \frac{3i}{4}$

$\cos(a)\cosh(b) - i\sin(a)\sinh(b) = 0+\frac{3i}{4}$ $\{\because \cos(ix)=\cosh(x), \sin(ix)=\sinh(x);\}$

Comparing Real and Imaginary terms on both sides,

$\cos(a)\cosh(b)=0$ ....(2) & $-\sin(a)\text{sinj}(b) = \frac{3}{4}$ ....(3)

From (2), $\cos(a)=0$ or $\cosh(b)=0$,

$\therefore a = \frac{\pi}{2}$ ....(4)

From (3) & (4), $-\sin\left(\frac{\pi}{2}\right)\sinh(b) = \frac{3}{4}$

$\therefore 1.\sinh(b) = \frac{-3}{4}$

$\therefore b = \sinh^{-1}\left(\frac{-3}{4}\right)$

$= \log\left[\left(\frac{-3}{4}\right) + \sqrt{\left(\frac{-3}{4}\right)^2 + 1}\right]$ $\{\because sinh^{-1}z = log\left(z + \sqrt{z^2+1}\right)\}$

$=\log\left[\left(\frac{-3}{4}\right) + \sqrt{\frac{9}{16} + 1}\right]$

$=\log\left[\left(\frac{-3}{4}\right)+\frac{5}{4}\right]$

$=\log\frac{1}{2}$

$=\log 2^{-1}$

$\therefore$ b = -log2 ....(5)

Substituting (4) & (5) in (1), $\cos^{-1}\left(\frac{3i}{4}\right)=\frac{\pi}{2}-i\log 2$

Comparing Real and Imaginary terms on both sides,

Real part = $a=\frac{\pi}{2}$
Imaginary part = b = -log2

1b) Show that the matrix A is unitary where A = $\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta\\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}$ is unitary if

$\alpha^2+\beta^2+\gamma^2+\delta^2=1$ **(3 marks)**

Ans: $A=\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta\\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}$

$\therefore A^T=\begin{bmatrix}\alpha+i\gamma & \beta+i\delta\\ -\beta+i\delta & \alpha-i\gamma\end{bmatrix}$

$\therefore A^\theta=\overline{A^T}=\begin{bmatrix}\alpha-i\gamma & \beta-i\delta\\ -\beta-i\delta & \alpha+i\gamma\end{bmatrix}$

Given, A is Unitary

$\therefore AA^\theta=1$

$\therefore \begin{bmatrix}\alpha+i\gamma & -\beta+i\delta\\ \beta+i\delta & \alpha-i\gamma\end{bmatrix}\times\begin{bmatrix}\alpha-i\gamma & \beta-i\delta\\ -\beta-i\delta & \alpha+i\gamma\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\therefore \begin{bmatrix}(\alpha+i\gamma)(\alpha-i\gamma)+(-\beta+i\delta)(-\beta-i\delta) & (\alpha+i\gamma)(\beta-i\delta)+(-\beta+i\delta)(\alpha+i\gamma)\\ (\beta+i\delta)(\alpha-i\gamma)+(\alpha-i\gamma)(-\beta-i\delta) & (\beta+i\delta)(\beta-i\delta)+(\alpha-i\gamma)(\alpha+i\gamma)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ ...(1)

Consider,

$(\alpha + i\gamma)(\alpha - i\gamma) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2 - i^2\gamma^2 + \beta^2 - i^2\delta^2$

$(\alpha + i\gamma)(\alpha - i\gamma) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2+\beta^2+\gamma^2+\delta^2$ ...(2)

And, $(\alpha + i\gamma)(\beta - i\delta) + (-\beta + i\delta)(\alpha + i\gamma) = (\alpha\beta - i\alpha\delta + i\beta\gamma - i^2\gamma\delta) + (-\alpha\beta - i\beta\gamma + i\alpha\delta + i^2\gamma\delta) = 0$ ...(3)

Similarlly,

$(\beta + i\delta)(\alpha - i\gamma) + (\alpha - i\gamma)(-\beta - i\delta) = 0$ ...(4)

$(\beta + i\delta)(\beta - i\delta) + (\alpha - i\gamma)(\alpha + i\gamma) = \alpha^2+\beta^2+\gamma^2+\delta^2$ ...(5)

Substituting (2),(3),(4)&(5) in (1),

$$\begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing corresponding terms, we get,

$$\boxed{\alpha^2+\beta^2+\gamma^2+\delta^2=1}$$

1c) If $z = \tan(y + ax) + (y - ax)^{3/2}$ then show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$ **(3 marks)**

Ans: $z = \tan(y + ax) + (y - ax)^{3/2}$ ...(1)

Differentiate partially w.r.t.x, $\frac{\partial z}{\partial x} = \sec^2(y + ax)\cdot a + \frac{3}{2}(y - ax)^{1/2}\cdot(-a)$

$\therefore \frac{\partial z}{\partial x} = a\sec^2(y + ax) - \frac{3a}{2}(y - ax)^{1/2}$

Again, differentiate partially w.r.t.x,

$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y + ax)\cdot \tan(y + ax) - \frac{3}{4}a^2(y - ax)^{-1/2}$

$\therefore \frac{\partial^2 z}{\partial x^2} = a^2\left[2sec^2(y + ax)\tan(y + ax) - \frac{3}{4}(y - ax)^2\right]$ ...(2)

Differentiate (1) partially w.r.t.y, $\frac{\partial z}{\partial y} = \sec^2(y + ax)\cdot 1 + \frac{3}{2}(y - ax)^{1/2}$

Again, differentiate partially w.r.t.y, $\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y + ax)\cdot \tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2}$ ...(3)

From (2)&(3), $\boxed{\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}}$

1d) If $x=uv$, $y=\frac{u}{v}$. Prove that $JJ'=1$. **(3marks)**

Ans: $x=uv$ ...(1)

$\therefore x_u = \frac{\partial x}{\partial u} = v$ and $x_v = \frac{\partial x}{\partial v} = u$ ...(2)

And, $y = \frac{u}{v}$ ...(3)

$\therefore y_u = \frac{\partial y}{\partial u} = \frac{1}{v}$ and $y_v = \frac{\partial y}{\partial v} = u\frac{-1}{v^2}$ ...(4)

$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= x_u y_v - x_v y_u$

$= v\,u\frac{-1}{v^2} - u\frac{1}{v}$ ...(From 2 & 4)

$= \frac{-u}{v} - \frac{u}{v}$

$= \frac{-2u}{v}$

$\therefore J = -2y$ ...(5)

From (3), $u = vy$ ...(6)

Substituting 'u' in (1) we get, $x= (vy)v$

$\frac{x}{y} = v^2$

$\therefore v = \frac{\sqrt{x}}{\sqrt{y}} = x^{1/2}y^{1/2}$ ...(7)

$\therefore v_x = y^{-1/2}.\frac{1}{2}\,x^{-1/2}$ and $v_y = x^{1/2}.\frac{-1}{2}\,y^{-3/2}$ ...(8)

From (6) and (7), $u = (x^{1/2}y^{-1/2})y$

$\therefore u = x^{1/2}y^{1/2}$

$\therefore$ $u_x = y^{1/2}$. and $u_y = x^{1/2} . \frac{1}{2} y^{-1/2}$ ...(9)

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= u_x v_y - u_y v_x$

$= \left(y^{1/2} \cdot \frac{1}{2} x^{-1/2}\right)\left(x^{1/2} \cdot \frac{-1}{2} y^{-3/2}\right) - \left(x^{1/2} \cdot \frac{1}{2} y^{-1/2}\right)\left(y^{-1/2} \cdot \frac{1}{2} x^{-1/2}\right)$ ...(From 8 & 9)

$= \frac{-1}{4} x^{\frac{-1}{2}+\frac{1}{2}} \cdot y^{\frac{1}{2}-\frac{3}{2}} - \frac{-1}{4} x^{\frac{1}{2}-\frac{1}{2}} \cdot y^{\frac{-1}{2}-\frac{1}{2}}$

$= \frac{-1}{4} \cdot y^{-1} - \frac{1}{4} \cdot y^{-1}$

$= \frac{-2}{4} \cdot y^{-1}$

$\therefore J' = \frac{-1}{2y}$ ...(10)

From (5) and (10), $J \cdot J' = -2y \cdot \frac{-1}{2y}$

$\therefore J \cdot J' = 1$

1e) Find the n^th derivative of $\frac{x^3}{(x+1)(x-2)}$. **(4 marks)**

Ans: Let $y = \frac{x^3}{(x+1)(x-2)} = \frac{x^3}{x^2-x-2}$

Consider, $x^2 - x - 2 \,\overline{)\, x^3 + 0x^2 + 0x + 0}$ quotient $x + 1$

$\underline{x^3 - x^2 - 2x}$

$x^2 + 2x + 0$

$\underline{x^2 - x - 2}$

$3x + 2$

$\therefore$ $y = x + 1 + \frac{3x+2}{x^2-x-2}$

$\therefore$ y = x + 1 + $\frac{3x+2}{(x+1)(x-2)}$

$\therefore$ y = x + 1 + $\frac{1/3}{(x+1)}+\frac{8/3}{(x-2)}$ (By Partial Fraction)

Taking n[th] order derivative, $y_n = 0+0+\frac{1}{3}\cdot\frac{n!\cdot 1^n\,(-1)^n}{(x+1)^{n+1}}+\frac{8}{3}\cdot\frac{n!\cdot 1^n\,(-1)^n}{(x-2)^{n+1}}$

$$\left\{\text{If y} = \frac{1}{ax+b}\ then\ y_n = \frac{n!a^n(-1)^n}{(ax+b)^{n+1}}\right\}$$

$$\therefore y_n = \frac{n!\,(-1)^n}{3}\left[\frac{1}{(x+1)^{n+1}}+\frac{8}{(x-2)^{n+1}}\right]$$

1f) Using the matrix A = $\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}$ decode the message of matrix C = $\begin{bmatrix}4 & 11 & 12 & -2\\ -4 & 4 & 9 & -2\end{bmatrix}$

(4 marks)

Ans: Encoding Matrix A = $\begin{bmatrix}-1 & 2\\ -1 & 1\end{bmatrix}$ ...(1)

Given, C = $\begin{bmatrix}4 & 11 & 12 & -2\\ -4 & 4 & 9 & -2\end{bmatrix}$

Step 1:

Writing the numbers in C matrix column wise gives the encoded message.

$\therefore$ Encoded Message = 4 -4 11 4 12 9 -2 -2

This Encoded message is transmitted.

Assume there is no corruption of data, the message at the receiving end is 4 -4 11 4 12 9 -2 -2

This message is decoded

Step 2;

We know, if P = $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then $P^{-1} = \frac{1}{|P|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

From (1), |A| = -1 + 2 = 1 ...(2)

$\therefore$ Decoding matrix $A^{-1} = \frac{1}{1}\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}$ (From 2) ...(3)

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From (2) & (3), $A^{-1}C = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 11 & 12 & -2 \\ -4 & 4 & 9 & -2 \end{bmatrix}$

$\therefore A^{-1}C = \begin{bmatrix} 4+8 & 11-8 & 12-18 & -2+4 \\ 4+4 & 11-4 & 12-9 & -2+2 \end{bmatrix}$

$\therefore A^{-1}C = \begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}$

Step 3:

Considering the numbers column-wise we get,

12 8 3 7 -6 3 2 0

Decoded Message = 12 8 3 7 -6 3 2 0 or $\begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}$

Q.2) **(20 marks)**

2a) If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ then find a,b,c,d. **(6 marks)**

Ans: We know, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ ...(1)

Consider, $\sin^4\theta\cos^3\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \times \frac{e^{i\theta}+e^{-i\theta}}{2}$ (From 1)

$= \frac{1}{2^4 i^4 2^3} \times (e^{i\theta} - e^{-i\theta})(e^{i\theta} - e^{-i\theta})^3(e^{i\theta} + e^{-i\theta})^3$

$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta})\left[(e^{i\theta})^2 - (e^{-i\theta})^2\right]^3$

$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta})[(e^{2i\theta}) - (e^{-2i\theta})]^3$

$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta})\left[(e^{2i\theta})^3 - 3(e^{2i\theta})^2(e^{-2i\theta}) + 3(e^{2i\theta})(e^{-2i\theta})^2 - (e^{-2i\theta})^3\right]$

$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta})\left[e^{6i\theta} - 3e^{2i\theta} + 3e^{-2i\theta} - e^{-6i\theta}\right]$

$= \frac{1}{2^7} \times \left[e^{7i\theta} - 3e^{3i\theta} + 3e^{-i\theta} - e^{-5i\theta} - e^{5i\theta} + 3e^{i\theta} - 3e^{-3i\theta} + e^{-7i\theta}\right]$

$= \frac{1}{2^7}\left[(e^{7i\theta} + e^{-7i\theta}) - (e^{5i\theta} + e^{-5i\theta}) - 3(e^{3i\theta} + e^{-3i\theta}) + 3(e^{i\theta} + e^{-i\theta})\right]$

$= \frac{1}{128} \times 2\cos7\theta - \frac{1}{128} \times 2\cos5\theta - \frac{1}{128} \times 6\cos3\theta + \frac{1}{128} \times 6\cos\theta$

$\therefore \sin^4\theta\cos^3\theta = \frac{3}{64}\cos\theta - \frac{3}{64}\cos3\theta - \frac{1}{64}\cos5\theta + \frac{1}{64}\cos7\theta$ ...(2)

But , given, $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ ...(3)

Comparing (2) & (3), $a = \frac{3}{64}$; $b = \frac{-3}{64}$; $c = \frac{-1}{64}$; $d = \frac{1}{64}$;

2b) Using Newton Raphson method solve 3x – cosx – 1 = 0. Correct upto 3 decimal places.

(6 marks)

Ans: Let $f(x) = 3x - \cos x - 1$

$\therefore f'(x) = 3 + \sin x - 0$

When x = 0, $f(0) = 3(0) - \cos0 - 1 = -2$

When x =1, $f(1) = 3(1) - \cos1 - 1 = 1.4597$

$\therefore$ Roots of $f(x)$ lies between 0 and 1.

Let initial value $x_0 = 0$

By Newton-Raphson's Method $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$

$= \frac{x_n(3+\sin x_n) - (3x_n - \cos x_n - 1)}{3+\sin x_n}$

$= \frac{3x_n + x_n\sin x_n - 3x_n + \cos x_n + 1}{3+\sin x_n}$

$\therefore x_{n+1} = \frac{x_n\sin x_n + \cos x_n + 1}{3+\sin x_n}$ ...(1)

Iteration 1: Put n = 0 in (1)

$\therefore x_1 = \frac{x_0\sin x_0 + \cos x_0 + 1}{3+\sin x_0} = \frac{0+\cos0+1}{3+\sin0} = 0.6667$

Iteration 2: Put n = 1 in (1)

$\therefore x_2 = \frac{x_1\sin x_1 + \cos x_1 + 1}{3+\sin x_1} = \frac{0.6667\sin(0.6667)+\cos(0.6667)+1}{3+\sin(0.6667)} = 0.6075$

<u>Iteration 3:</u> Put n = 2 in (1)

$\therefore x_3 = \frac{x_2 sinx_2 + cosx_2 + 1}{3 + sinx_2} = \frac{0.6075 sin(0.6075) + cos(0.6075) + 1}{3 + sin(0.6075)} = 0.6071$

<u>Iteration 4:</u> Put n = 3 in (1)

$\therefore x_4 = \frac{x_3 sinx_3 + cosx_3 + 1}{3 + sinx_3} = \frac{0.6071 sin(0.6071) + cos(0.6071) + 1}{3 + sin(0.6071)} = 0.6071$

Hence, Root of 3x – cosx – 1 = 0 is 0.6071

2c) Find the stationary points of the function $x^3+3xy^2-3x^2-3y^2+4$ & also find maximum and minimum values of the function. (8 marks)

Ans: Let $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ ...(1)

$\therefore f_x = 3x^2 + 3y^2 - 6x - 0 + 0$

$\therefore r = f_{xx} = 6x - 6$ ...(2)

Also, $f_y = 0 + 6xy - 0 - 6y + 0$

$\therefore t = f_{yy} = 6x - 6$ ...(3)

$\therefore s = f_{xy} = 0 + 6y - 0$ ...(4)

Put $f_x = 0$ and $f_y = 0$

$\therefore 3x^2 + 3y^2 - 6x = 0$

$\therefore x^2 + y^2 - 2x = 0$ ...(5)

And, $6xy - 6y = 0$

$\therefore 6y(x-1) = 0$

$\therefore y = 0$ or $x = 1$

<u>Case I</u> : When x = 1

From (5), $1^2 + y^2 - 2(1) = 0$

$\therefore y^2 - 1 = 0$

$\therefore y = \pm 1$

Case II: When y = 0

From (5), $x^2 + 0 - 2x = 0$

$\therefore x(x-2) = 0$

$\therefore x = 0$ or $x = 2$

$\therefore$ Stationary points are (1,1);(1,-1);(0,0);(2,0):

(i)At (1,1)

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at (1,1)

(ii)At (1,-1)

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at (1,-1)

(iii)At (0,0)

From (2), $r = 6(0) - 6 = -6 < 0$

From (3), $t = 6(0) - 6 = -6$

From (4), $s = 6(0) = 0$

$\therefore rt - s^2 = (-6)(-6) - 0 = 36 > 0$

$\therefore f$ has maximum at (0,0)

From (1), Maximum value of f

$\therefore f = (0)^3 + 3(0)(0)^2 - 3(0)^2 - 3(0)^2 + 4 = 4$

(iv)At (2,0)

From (2), r = 6(2) – 6 = 6 < 0

From (3), t = 6(2) – 6 = 6

From (4), s = 6(0) = 0

$\therefore$ rt – s^2 = (6)(6) – 0 = 36 > 0

$\therefore$ f has maximum at (2,0)

From (1), Minimum value of f

$\therefore f = (2)^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4 = 0$

Hence the function has

Maximum at (0,0) and Maximum value = 4
Minimum at (2,0) and Minimum value = 0

Q.3) (20 marks)

3a) Show that $\boldsymbol{x\,cosec\,x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots}$ **. (6 marks)**

Ans: LHS = $x\,cosec\,x$

$$= \frac{x}{\sin x}$$

$$= \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

$$= \frac{x}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}$$

$$= \left[\left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right)\right)\right]^{-1}$$

$$= 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\right)^2 + \cdots \quad \{\because (1-y)^{-1} = 1 + y + y^2 + y^3 + \cdots\}$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2 + \cdots$$

$= 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$

$$\therefore x cosec x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$$

3b) Reduce matrix to PAQ normal form and find 2 non-Singular matrices P & Q.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(6 marks)

Ans: $A_{3x4} = I_{3x3} \times A_{3x4} \times I_{4x4}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1$; $R_3 - R_1$; $\rightarrow$ $$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1$; $C_3 + C_1$; $C_4 - 2C_1$; $\rightarrow$ $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 + C_2$; $\frac{1}{2} C_3$ $\rightarrow$ $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LHS is the required PAQ form .

Here, $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3c) If y=cos(msin^{-1}x). Prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$. **(8 marks)**

Ans: $y = \cos(m\sin^{-1}x)$ ...(1)

Differentiating w.r.t. 'x', $y_1 = -\sin(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$

$\therefore \sqrt{1-x^2}\cdot y_1 = -m\sin(m\sin^{-1}x)$

On Squaring, $(1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x)$

$\therefore (1-x^2)y_1^2 = m^2\,[1 - \cos^2(m\sin^{-1}x)]$

$\therefore (1-x^2)y_1^2 = m^2\,[1 - y^2]$ (From 1)

Again differentiating w.r.t. 'x', $(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2(0 - 2yy_1)$

$\therefore (1-x^2)y_2 - xy_1 = -m^2y$ (Dividing by $2y_1$)

Applying Leibnitz theorem, $\{y_n = u_nv + nu_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + {}^nC_3u_{n-3}v_3 + \cdots\}$

$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}\right] - [xy_{n+1} + ny_n] = -m^2y_n$

$\therefore (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!} - xy_{n+1} - ny_n + m^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - xy_n(2n+1) + (-n^2 + n - n + m^2)y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$

Q.4) **(20 marks)**

4a) State and Prove Euler's Theorem for three variables. **(6 marks)**

Ans: Euler's theorem:

Statement: If 'u' is a homogenous function of three variables x, y, z of degree 'n' then Euler's theorem

States that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Proof:

Let $u = f(x, y, z)$ be the homogenous function of degree 'n'.

Let X = xt, Y = yt, Z = zt

$\therefore \frac{\partial X}{\partial t} = x;\ \frac{\partial Y}{\partial t} = y;\ \frac{\partial Z}{\partial t} = z$ ...(1)

At t = 1, ...(2)

X = x, Y = y, Z = z

$\therefore \frac{\partial f}{\partial X} = \frac{\partial f}{\partial x};\ \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial y};\ \frac{\partial f}{\partial Z} = \frac{\partial f}{\partial z};$ ...(3)

Now, $f(X, Y, Z) = t^n f(x, y, z)$ ...(4)

$\therefore f \rightarrow X, Y, Z \rightarrow x, y, z, t$

Differentiating (4) partially w.r.t. 't', $\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial t} = nt^{n-1}f(x,y,z)$

$\therefore \frac{\partial f}{\partial X}\cdot x + \frac{\partial f}{\partial Y}\cdot y + \frac{\partial f}{\partial Z}\cdot z = n(1)^{n-1}f(x,y,z)$ (From 1,2 & 3)

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

4b) Show that all roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by $-i\cot\frac{(2k+1)\pi}{12}$ where k=0,1,2,3,4,5.

(6 marks)

Ans: $(x + 1)^6 + (x - 1)^6 = 0$

$\therefore (x + 1)^6 = -(x - 1)^6$

$\therefore \frac{(x+1)^6}{(x-1)^6} = -1$

$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i\pi}$ $\{\because e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1\}$ (Principal value)

$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i(\pi+2k\pi)}$, k= 0, 1, 2, 3, 4, 5 (General Value)

$\therefore \frac{x+1}{x-1} = e^{i\pi(1+2k)/6}$ ...(1)

Let $2\theta = \frac{\pi(1+2k)}{6}$ ...(2)

$\therefore$ From (1) & (2), $\frac{x+1}{x-1} = e^{i2\theta}$

$\therefore$ By Componendo – Dividendo, $\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i2\theta}+1}{e^{i2\theta}-1}$

$\therefore \quad \frac{2x}{2} = \frac{e^{i\theta}[e^{i\theta}+e^{-i\theta}]}{e^{i\theta}[e^{i\theta}-e^{-i\theta}]}$

$\therefore \quad x = \frac{2cos\theta}{2isin\theta}$ $\quad \left\{\because sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \ and \ cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right\}$

$\therefore$ x = $\frac{1}{i}$ cotθ

$\therefore$ x = -icot$\frac{(2k+1)n}{12}$ (From 2) where k = 0, 1, 2, 3, 4, 5

4c) Show that the following equations: -2x + y + z = a, x - 2y + z = b, x + y - 2z = c have no solutions unless a +b + c = 0 in which case they have infinitely many solutions. Find these solutions when a=1, b=1, c=-2. (8 marks)

Ans: Part I:

-2x + y + z = a

x - 2y + z = b

x + y - 2z = c

Writing the equations in the matrix form,

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$R_3+(R_1+R_2) \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b+c \end{bmatrix}$ ...(1)

Augmented matrix [A|B] = $\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \end{array}\right]$

Number of unknowns = n = 3

Rank of A (r_A) = Number of non-zero rows in A = 2

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when 'a+b+c≠0' upon which,

Rank of [A|B] = $(r_{AB}) = 3$

Case II: Infinite Solution

For which, $r_A = r_{AB} < n$ (i.e. < 3)

This is only possible, when 'a+b+c=0' upon which,

Rank of [A|B] = $r_{AB} = 2$

Part II: Put a= 1, b = 1, c = -2, in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$R_2 - R_1$; → $$\begin{bmatrix} -2 & 1 & 1 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(2)}$$

Here, $n - r_A = 3 - 2 = 1$

We have to assume one unknown.

Let y = t (≠0)

On expanding (2), $3x - 3y = 0$

$\therefore x - y = 0$

$\therefore x = y = t$

And, $-2x + y + z = 1$

$\therefore -2t + t + z = 1$

$\therefore z = 1 + t$

Hence, the solution is

x = t, y = t, z = 1 + t (Infinite Solution)

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Q5)

5a) If z = f(x. y). x = r cos θ, y = r sin θ. prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

(6 marks)

Ans: x = r cos θ and y = r sin θ ...(1)

Differentiating partially w.r.t. 'θ'. $\frac{\partial x}{\partial \theta}$ = - rsinθ; $\frac{\partial y}{\partial \theta}$= rcosθ; ...(2)

Differentiating partially w.r.t.' r', $\frac{\partial x}{\partial r}$=cosθ; $\frac{\partial y}{\partial r}$=sinθ ...(3)

Now, z → x, y →r , θ

By Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$

∴ $\frac{\partial z}{\partial r}$ = cosθ $\frac{\partial z}{\partial x}$ + sinθ $\frac{\partial z}{\partial y}$ (From 3) ...(4)

Similarly, By Chain Rule, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$

∴ $\frac{\partial z}{\partial \theta}$ = - rsinθ $\frac{\partial z}{\partial x}$ + rcosθ $\frac{\partial z}{\partial y}$ (From 2) ...(5)

RHS = $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

= $\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\,\frac{\partial z}{\partial y}\right)^2$ (From 4 & 5)

= $cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2sin\theta\frac{\partial z}{\partial x}\cdot cos\theta\frac{\partial z}{\partial y} + sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2 - 2sin\theta\frac{\partial z}{\partial x}\,cos\theta\frac{\partial z}{\partial y} + cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$

= $\left(\frac{\partial z}{\partial x}\right)^2 (cos^2\theta + sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2 (cos^2\theta + sin^2\theta)$

= $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

= LHS

Hence, $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

5b) If coshx = secθ prove that (i) x = log(secθ+tanθ). (ii) θ=$\frac{\pi}{2}$tan^{-1}(e^{-x}) **(6 marks)**

Ans: (i) coshx = secθ

$$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta \qquad \left\{\because \cosh x = \frac{e^x+e^{-x}}{2}\right\}$$

$$\therefore e^x + \frac{1}{e^x} = 2sec\theta$$

$$\therefore (e^x)^2 + 1 = 2sec\theta e^x$$

$$\therefore (e^x)^2 - 2sec\theta e^x + 1 = 0$$

$$\therefore e^x = \frac{2sec\theta \pm \sqrt{4sec^2\theta - 4}}{2} \qquad \left\{\because Using, x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right\}$$

$$\therefore e^x = \frac{2sec\theta \pm \sqrt{4(sec^2\theta - 1)}}{2}$$

$$\therefore e^x = \frac{2sec\theta \pm 2tan\theta}{2}$$

$$\therefore e^x = sec\theta \pm tan\theta$$

Considering only positive root,

$$\therefore e^x = sec\theta + tan\theta \qquad \ldots(1)$$

$$\therefore \text{x} = \log(\sec\theta + \tan\theta)$$

(ii) From (1), $e^x = \frac{1}{cos\theta} + \frac{sin\theta}{cos\theta}$

$$\therefore e^x = \frac{1+sin\theta}{cos\theta}$$

$$\therefore \frac{1}{e^x} = \frac{cos\theta}{1+sin\theta}$$

$$\therefore e^{-x} = \frac{sin\left(\frac{\pi}{2}-\theta\right)}{1+cos\left(\frac{\pi}{2}-\theta\right)}$$

Put $\alpha = \frac{\pi}{2} - \theta$ ...(2)

$$\therefore e^{-x} = \frac{sin\alpha}{1+cos\alpha}$$

$$\therefore e^{-x} = \frac{2sin(\alpha/2)cos(\alpha/2)}{2cos^2(\alpha/2)}$$ $\{\because 2sinAcosA = sin2A; 1 + cos2A = 2cos^2A\}$

$$\therefore e^{x} = tan\left(\frac{\alpha}{2}\right)$$

$$\therefore tan^{-1}(e^{-1}) = \frac{\alpha}{2}$$

$$\therefore 2tan^{-1}(e^{-1}) = \alpha$$

$$\therefore 2tan^{-1}(e^{-1}) = \frac{\pi}{2} - \theta$$ (From 2)

$$\boxed{\therefore \theta = \frac{\pi}{2} - 2tan^{-1}(e^{-1})}$$

5c) Solve by Gauss Jacobi Iteration Method: 5x – y + z = 10, 2x + 4y = 12, x + y + 5z = -1.

(8 marks)

Ans: From 1st equation, 5x = 10 + y – z

$\therefore x = \frac{1}{5}(10 + y - z) = 0.2(10 + y - z)$

Similarly,

From 2nd equation, x + 2y = 6

$\therefore 2y = 6 - x$

$y = \frac{1}{2}(6 - x) = 0.5(6 - x)$ and,

$z = 0.2(-1 - x - y) = -0.2(1 + x + y)$

<u>Iteration 1:</u>

Put $x_0 = y_0 = z_0$

$\therefore x_1 = = 0.2(10 + y_0 - z_0) = = 0.2(10 + 0 - 0) = 2$

$\therefore y_1 = 0.5(6 - x_0) = 0.5(6 - 0) = 3$

$\therefore z_1 = -0.2(1 + x_0 + y_0) = -0.2(1 + 0 + 0) = -0.2$

<u>Iteration 2:</u>

Put $x_1 = 2; y_1 = 3; z_1 = -0.2$

$\therefore x_2 = 0.2(10 + y_1 - z_1) = 0.2(10 + 3 + 0.2) = 2.64$

$\therefore y_2 = 0.5(6 - x_1) = 0.5(6 - 2) = 2$

$\therefore z_2 = -0.2(1 + x_1 + y_1) = -0.2(1 + 2 + 3) = -1.2$

<u>Iteration 3:</u>

Put $x_2 = 2.64; y_2 = 2; z_2 = -1.2$

$\therefore x_3 = 0.2(10 + y_2 - z_2) = 0.2(10 + 2 = 1.2) = 2.64$

$\therefore y_3 = 0.5(6 - x_2) = 0.5(6 - 2.64) = 1.68$

$\therefore z_3 = -0.2(1 + x_2 + y_2) = -0.2(1 + 2.64 + 2) = -1.128$

<u>Iteration 4:</u>

Put $x_3 = 2.64; y_3 = 1.68; z_3 = -1.128$

$\therefore x_4 = 0.2(10 + y_3 - z_3) = 0.2(10 + 1.68 = 1.128) = 2.5615$

$\therefore y_4 = 0.5(6 - x_3) = 0.5(6 - 2.64) = 1.68$

$\therefore z_4 = -0.2(1 + x_3 + y_3) = -0.2(1 + 2.64 + 1.68) = -1.0640$

<u>Iteration 5:</u>

Put $x_4 = 2.5616; y_4 = 1.68; z_4 = -1.0640$

$\therefore x_5 = 0.2(10 + y_4 - z_4) = 0.2(10 + 1.68 + 1.0640) = 2.5488$

$\therefore y_5 = 0.5(6 - x_4) = 0.5(6 - 2.5616) = 1.7172$

$\therefore z_5 = -0.2(1 + x_4 + y_4) = -0.2(1 + 2.5616 + 1.68) = -1.0483$

<u>Iteration 6:</u>

Put $x_5 = 2.5488; y_5 = 1.7192; z_5 = -1.0483$

$\therefore x_6 = 0.2(10 + y_5 - z_5) = 0.2(10 + 1.7192 + 1.0483) = 2.5535$

$\therefore y_6 = 0.5(6 - x_5) = 0.5(6 - 2.5488) = 1.7256$

$\therefore z_6 = -0.2(1 + x_5 + y_5) = -0.2(1 + 2.5488 + 1.7192) = -1.0536$

<u>Iteration 7:</u>

Put $x_6 = 2.5535; y_6 = 1.7256; z_6 = -1.0536$

$\therefore x_7 = 0.2(10 + y_6 - z_6) = 0.2(10 + 1.7256 =1.0536) = 2.5558$

$\therefore y_7 = 0.57(6 - x_6) = 0.5(6 - 2.5535) = 1.7232$

$\therefore z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5535 + 1.7256) = -1.0558$

<u>Iteration 8:</u>

Put $x_7 = 2.5558; y_7 = 1.7232; z_7 = -1.0558$

$\therefore x_8 = 0.2(10 + y_7 - z_7) = 0.2(10 + 1.7232 =1.0558) = 2.5558$

$\therefore y_8 = 0.57(6 - x_7) = 0.5(6 - 2.5558) = 1.7221$

$\therefore z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5558 + 1.7232) = -1.0558$

Hence, by Gauss Jacobi Iteration Method, the solution is

$x = 2.5558, y = 1.7221, z = -1.0558$

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Q.6) **(20 marks)**

6a) Prove that $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)$ **(6 marks)**

Ans: We know, $\tanh\theta = \frac{sinh\theta}{cosh\theta}$

$= \frac{(e^{\theta} - e^{-\theta})/2}{(e^{\theta} + e^{-\theta})/2}$

$\therefore$ $\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$

Put θ=log x, ...(1)

$\therefore$ $\tanh(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$

$= \frac{e^{\log x} - e^{\log x^{-1}}}{e^{\log x} + e^{\log x^{-1}}}$

$= \frac{x - x^{-1}}{x + x^{-1}}$

$= \frac{x(1 - x^{-2})}{x(1 + x^{-2})}$ ...(2)

Let y = $\cos^{-1}[\tanh(\log x)]$

$= \cos^{-1}\left[\frac{1 - x^{-2}}{1 + x^{-2}}\right]$ (From 2)

$= \cos^{-1}\left[\frac{1 - (x^{-1})^{-2}}{1 + (x^{-1})^{-2}}\right]$

Put $x^{-1} = \tan\theta$

$\therefore$ $y = \cos^{-1}\left(\frac{1 - tan^2\theta}{1 + tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$= 2\tan^{-1}\left(\frac{1}{x}\right)$ (From 1)

$= 2\cot^{-1}x$

$= 2\left(\frac{\pi}{2} - \tan^{-1}x\right)$

$= \pi - 2\tan^{-1}x$

$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\right)$

Hence, $\boxed{\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\right)}$

6b) If y = $e^{2x}\sin\frac{x}{2}\cos\frac{x}{2}\sin 3x$. Find y_n. **(6 marks)**

Ans: y = $e^{2x}\sin\frac{x}{2}\cos\frac{x}{2}\sin 3x \times \frac{2}{2}$

$= \frac{1}{2}e^{2x}\left[\sin 2\left(\frac{x}{2}\right)\right]\sin 3x \qquad \{\because 2\sin A\cos A = \sin 2A\}$

$= \frac{1}{2}e^{2x}\sin x\sin 3x \times \frac{2}{2}$

$= \frac{1}{4}e^{2x}[\cos(3x - x) - \cos(3x + x)] \qquad \{\because 2\sin A\sin B = \cos(A - B) - \cos(A + B)\}$

$\therefore$ y = $\frac{1}{4}[e^{2x}\cos 2x - e^{2x}\cos 4x]$

Taking n^{th} order derivative, $y_n = \frac{1}{4}\left\{\frac{d^n}{dx^n}(e^{2x}\cos 2x) - \frac{d^n}{dx^n}(e^{2x}\cos 4x)\right\}$ ...(1)

We know, If y = $e^{ax}\cos(bx + c)$, $y_n = r^n e^{ax}\cos(bx + c + n\emptyset)$ ...(2)

Here a = 2, c = 0, $b_1 = 2$ and $b_2 = 4$

$\therefore r_1 = \sqrt{a^2 + b_1^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 8^{1/2}$ and $r_2 = \sqrt{a^2 + b_2^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 20^{1/2}$ ...(3)

And, $\emptyset_1 = \tan^{-1}\frac{b_1}{a} = \tan^{-1}\frac{2}{2} = \tan^{-1}1 = \frac{\pi}{4}$ & $\emptyset_2 = \tan^{-1}\frac{b_2}{a} = \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$ ...(4)

$\therefore$ From (1),(2),(3) and (4),

$Y_n = \frac{1}{4}\left\{\left(8^{1/2}\right)^n e^{2x}\cos(2x + 0 + n\emptyset_1) + \left(20^{1/2}\right)^n e^{2x}\cos(4x + 0 + n\emptyset_2)\right\}$

$\boxed{\therefore y = \frac{1}{4}e^{2x}\left[8^{n/2}\cos\left(2x + \frac{n\pi}{4}\right) + 20^{n/2}\cos(4x + n\emptyset_2)\right]}$, where $\emptyset_2 = \tan^{-1}\frac{1}{2}$

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6c) (i) Evaluate $\lim_{x\to 0}(\cot x)^{\sin x}$. **(4 marks)**

(ii) Prove that $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x \tanh y)$. **(4 marks)**

Ans: (i) Let $L = \lim_{x\to 0}(\cot x)^{\sin x}$

$\therefore \log L = \log\{\lim_{x\to 0}(\cot x)^{\sin x}\}$

$= \lim_{x\to 0}\{\log(\cot x)^{\sin x}\}$

$= \lim_{x\to 0} \sin x \cdot \log(\cot x)$

$= \lim_{x\to 0} \frac{\log(\cot x)}{\operatorname{cosec} x}$ $\quad\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to 0} \frac{\frac{1}{\cot x}\cdot -\operatorname{cosec}^2 x}{-\operatorname{cosec} x \cot x}$ (L' Hospital's Rule)

$= \lim_{x\to 0} \tan x \cdot \frac{1}{\sin x} \cdot \tan x$

$= \lim_{x\to 0} \tan x \cdot \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x}$

$= \tan 0 \times \frac{1}{\cos 0}$

$\therefore \log L = 0$

$\therefore L = e^0$

$$\therefore \lim_{x\to 0}(\cot x)^{\sin x} = 1$$

(ii) Consider, log[sin(x+iy)]=log[sin x cos(iy) + cos x sin (iy)]

$\therefore$ log[sin(x+iy)]=log[sin x cosh y + icos x sinh y] $\{\because \cos(ix)=\cosh x;\ \sin(ix)=i\sinh x;\}$

$\therefore \log[\sin(x+iy)] = \frac{1}{2}\log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] + i\tan^{-1}\left|\frac{\cos x \sinh y}{\sin x \cosh y}\right|$

$$\left\{\because \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left|\frac{y}{x}\right|\right\}$$

$\therefore \log[\sin(x+iy)] = \frac{1}{2}\log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] + i\tan^{-1}|\cot x \tanh y|$

Taking Conjugate, $\log[\sin(x-iy)] = \frac{1}{2}\log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] - i\tan^{-1}|\cot x \tanh y|$

Now, $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x+iy)] - \log[\sin(x-iy)]$

$= \left\{\frac{1}{2}\log[\sin2\ x \cosh2\ y\ +\ \cos2\ x \sinh2\ y]\ +\ i\tan-1|\cot x \tanh y|\right\} -$

$\left\{\frac{1}{2}\log[\sin2\ x \cosh2\ y\ +\ \cos2\ x \sinh2\ y] -\ i\tan-1|\cot x \tanh y|\right\}$

$= 2i\tan^{-1}(\cot x \tanh y)$

$$\therefore \log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x \tanh y)$$

MUMBAI UNIVERSITY

SEMESTER 1

APPLIED MATHEMATICS SOLVED PAPER – MAY 2018

N.B:- (1) Question no. 1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1(a) If $\tan\frac{x}{2} = \tanh\frac{u}{2}$, show that $u = \log[(\tan(\frac{\pi}{4}+\frac{x}{2}))]$ **[3]**

Ans : Given that : $\tan\frac{x}{2} = \tanh\frac{u}{2}$

$$\frac{u}{2} = \tanh^{-1}[\tan\frac{x}{2}]$$

$$\therefore u = 2\tanh^{-1}[\tan\frac{x}{2}]$$

By using Inverse hyperbolic function,

$$= \log\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

But $\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{\frac{\pi}{4}+\tan\frac{x}{2}}{\frac{\pi}{4}-\tan\frac{x}{2}} = \tan(\frac{\pi}{4}+\frac{x}{2})$

$$\therefore u = \log[(\tan(\frac{\pi}{4}+\frac{x}{2}))]$$

Hence proved.

(b) Prove that the following matrix is orthogonal & hence find A^{-1}. **[3]**

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Ans : Let $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

Transpose of A is given by ,

$$A^T = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A.A^T = \frac{1}{9}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{I}{9}$$

The given matrix A is orthogonal.
The inverse of an orthogonal matrix is always equal to the Transpose of that particular matrix.

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

(c) State Euler's theorem on homogeneous function of two variables & if u = $\frac{x+y}{x^2+y^2}$ then evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ **[3]**

Ans : Euler's theorem : If a function 'u' is homogeneous with degree 'n' then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

Let $u = \frac{x+y}{x^2+y^2}$

Put x = xt and y = yt

$$F(x,y) = \frac{xt+yt}{(xt)^2+(yt)^2} = \frac{1}{t}\left[\frac{x+y}{x^2+y^2}\right]$$

$$= t^{-1} . f(u)$$

Hence the given function 'u' is homogeneous with degree n=-1

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\left[\frac{x+y}{x^2+y^2}\right]$$

(d) If u = $r^2 cos\ 2\theta, v = r^2 sin\ 2\theta$. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$. **[3]**

Ans : u = $r^2 cos\ 2\theta$ $\quad v = r^2 sin\ 2\theta$

Diff. u and v w.r.t r and θ partially to apply it in jacobian

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r cos\ 2\theta & -2r^2 sin\ 2\theta \\ 2r sin\ 2\theta & 2r^2 cos\ 2\theta \end{vmatrix}$$

$$= 4r^3 cos^2 2\theta + 4r^3 sin^2 2\theta$$

$$= 4r^3 (cos^2 2\theta + sin^2 2\theta)$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

(e) Find the nth derivative of cos 5x.cos 3x.cos x. **[4]**

Ans : let y = cos 5x.cos 3x.cos x

$$= \frac{cos\ (5x-3x)+cos\ (5x+3x)}{2} \text{ cos x } \left\{ \text{cos A.cos B} = \tfrac{1}{2}[cos\ (A+B) + COS\ (A-B)] \right\}$$

$$= \frac{1}{2}\ [\ cos\ 2x.\ cos\ x + cos\ 8x.\ cos\ x\ ]$$

$$y = \frac{1}{4}\ [\ cos\ 3x + cos\ x + cos\ 9x + cos\ 7x\ ]$$

Take n th derivative,

$$n\ th\ derivative\ of\ cos\ (ax+b) = a^n cos\ \left(\frac{n\pi}{2} + ax + b\right)$$

$$\boxed{y_n = \frac{1}{4}\ [3^n cos\left(\frac{n\pi}{2} + 3x\right) + cos\ \left(\frac{n\pi}{2} + x\right) + 9^n cos\ \left(\frac{n\pi}{2} + 9x\right) + 7^n cos\ \left(\frac{n\pi}{2} + 7x\right)]}$$

(f) Evaluate : $\lim_{x\to 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$ **[4]**

Ans : let $L = \lim_{x\to 0} \left(\frac{2x+1}{x+1}\right)^{\frac{1}{x}}$

Take log on both sides,

$\therefore \ \log L = \lim_{x\to 0} \frac{1}{x}\left(\frac{2x+1}{x+1}\right)$

$= \lim_{x\to 0}\left(\frac{2x+1}{x^2+x}\right)$

Apply L'Hospital rule ,

$\therefore \ \log L = \lim_{x\to 0}\left(\frac{2}{2x+1}\right)$

$\therefore \ \log L = 2$

$\therefore \ L = e^2$

Q. 2(a) Solve $x^4 - x^3 + x^2 - x + 1 = 0.$ [6]

Ans : $x^4 - x^3 + x^2 - x + 1 = 0$

Multiply the given eqn by (x+1),

$(x+1)(x^4 - x^3 + x^2 - x + 1)=0$

$x^5 = (-1)$

But $-1 = \cos\pi + i\sin\pi$

$\therefore \ x = [\cos\pi + i\sin\pi]^{1/5}$

But By De Moivres theorem ,

$(\cos x + i\sin x)^n = \cos nx + i\sin nx$

$\therefore \ x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

Add period $2k\pi$,

$\therefore \ x = \cos(1+2k)\frac{\pi}{5} + i\sin(1+2k)\frac{\pi}{5}$

Where k = 0,1,2,3,4.

The roots of given eqn is given by ,

Put k=0 $x_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = e^{\pi/5}$

k=1 $x_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} = e^{3\pi/5}$

k=2 $x_2 = cos\frac{\pi}{1} + i\,sin\frac{\pi}{1} = e^{\pi/1}$

k=3 $x_3 = cos\frac{7\pi}{5} + i\,sin\frac{7\pi}{5} = e^{7\pi/5}$

k=4 $x_4 = cos\frac{9\pi}{5} + i\,sin\frac{9\pi}{5} = e^{9\pi/5}$

The roots of eqn are : $e^{\pi/5}, e^{3\pi/5}, e^{\pi/1}, e^{7\pi/5}, e^{9\pi/5}$.

(b) If y=$e^{\tan^{-1}x}$. Prove that

$$(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$$ [6]

Ans : y = $e^{\tan^{-1}x}$ ....(1)

Diff. w.r.t x ,

$y_1 = e^{\tan^{-1}x}\frac{1}{x^2+1}$

$(x^2+1)y_1 = e^{\tan^{-1}x} = y$ --------(from 1)

Again diff. w.r.t x,

$(x^2+1)y_2 + 2xy_1 = y_1$ ....(1)

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$(uv)_n = u_n v + {}^n_1C u_{n-1}v_1 + {}^n_2C u_{n-2}v_2 + \cdots + uv_n$

$u = (x^2+1), v = y_2$ ...for first term in eqn (1)

$u = 2x\,, v = y_1$ ...for second term in eqn (1)

$\therefore\ (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n - y_{n+1} = 0$

$\therefore\ (1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$

Hence Proved.

(c) Examine the function $f(x,y) = xy(3-x-y)$ for extreme values & find maximum and minimum values of $f(x,y)$. [8]

Ans : $f(x,y) = xy(3-x-y) = 3xy - x^2y - xy^2$

Diff. function w.r.t x and y partially,

$\frac{\partial f(x,y)}{\partial x} = 3y - 2xy - y^2$ $\frac{\partial f(x,y)}{\partial y} = 3x - x^2 - 2xy$

$\frac{\partial f(x,y)}{\partial x} = 0$ $\frac{\partial f(x,y)}{\partial y} = 0$

$3y - 2xy - y^2 = 0$ & $3x - x^2 - 2xy = 0$

y=0 , 3-2x-y=0 & x=0 , 3-x-2y=0

Stationary points are : (0,0) ,(3,0) , (0,3) , (1,1)

$r = \frac{\partial^2 f}{\partial x^2} = -2y$, $t = \frac{\partial^2 f}{\partial y^2} = -2x$

$s = \frac{\partial^2 f}{\partial x \partial y} = 3 - 2x - 2y$

$s^2 = (3-2x-2y)^2$

$rt - s^2 = 4xy - (3-2x-2y)^2$

For point (0,0), $rt - s^2 = -9 < 0$

The point is of maxima .

For point (3,0), $rt - s^2 = -9 < 0$

The point is of maxima .

For (0,3), $rt - s^2 = -9 < 0$

The point is of maxima.

For point (1,1), $rt - s^2 = 3 > 0$

The point is of minima .

(a) **Maximum values : At (0,0) , (0,3), (3,0)**

At point <u>(0,0)</u> f(max)=0

At point (0,3) f(max)=0

At point (3,0) f(max)=0

(b) Minimum values : At (1,1)

At point (1,1) f(min)=1

The maximum and minimum values of function are 0 and 1.

Q.3(a) Investigate for what values of μ *and* λ the equation x+y+z=6; x+2y+3z=10; x+2y+λz=μ have

(i)no solution,

(ii) a unique solution,

(iii) infinite no. of solution. **[6]**

Ans : Given eqn : x+y+z=6 , x+2y+3z=10 , x+2y+λz=μ

A X = B

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\1 & 2 & 3 & 10\\1 & 2 & \lambda & \mu\end{array}\right]$

$R_2 - R_1,$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda-1 & \mu-6\end{array}\right]$$

$R_3 - R_2,$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda-3 & \mu-10\end{array}\right]$$

(i) When λ=3,$\mu \neq 10$ $then\ r(a) = 2, r(A \vdots B) = 3$

r(A)$\neq r(A \vdots B)$

Hence for λ=3 , $\mu \neq 10$ system is inconsistent.

No solution exist.

(ii) When λ≠3,$\mu \neq 10$,$r(A) = r(A \vdots B) = 3$
Unique solution exist.

(iii) When λ=3,$\mu = 10$ $r(A) = r(A \vdots B) = 2 < 3$
Infinite solution.

(b) If u = f($\frac{y-x}{xy}$, $\frac{z-x}{xz}$) , show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$. [6]

Ans : let u = f(r,s)

$\therefore$ r = $\frac{y-x}{xy}$ $\therefore$ s = $\frac{z-x}{xz}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{x^2} + \frac{\partial u}{\partial s}(\frac{-1}{x^2})$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\frac{(-1)}{y^2} + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(\frac{1}{z^2})$$

$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\boxed{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$$

Hence proved.

(c) Prove that log($\frac{a+ib}{a-ib}$)=2itan$^{-1}\frac{b}{a}$ & cos [i log($\frac{a+ib}{a-ib}$] = $\frac{a^2-b^2}{a^2+b^2}$ [8]

Ans : let L = log($\frac{a+ib}{a-ib}$)

Using logarithmic properties ,

L = log(a+ib) – log(a-ib)

= $\frac{1}{2}$ log($a^2 + b^2$) + i. $\tan^{-1}\frac{b}{a}$ – [$\frac{1}{2}$ log($a^2 + b^2$) - i. $\tan^{-1}\frac{b}{a}$]

$L = 2i\tan^{-1}\frac{b}{a}$

$\therefore \log(\frac{a+ib}{a-ib}) = 2i\tan^{-1}\frac{b}{a}$ **Hence Proved.**

$\therefore \frac{a+ib}{a-ib} = e^{2i\tan^{-1}\frac{b}{a}} = \cos(2\tan^{-1}\frac{b}{a}) + i\sin(2\tan^{-1}\frac{b}{a})$

$\frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib} = \cos(2\tan^{-1}\frac{b}{a}) + i\sin(2\tan^{-1}\frac{b}{a})$

$\frac{a^2-b^2}{a^2+b^2} + imaginary = \cos(2\tan^{-1}\frac{b}{a}) + i\sin(2\tan^{-1}\frac{b}{a})$

Separate real and imaginary parts

$\cos(2\tan^{-1}\frac{b}{a}) = \frac{a^2-b^2}{a^2+b^2}$

From 1st result ,

$\cos[i\log(\frac{a+ib}{a-ib})] = \frac{a^2-b^2}{a^2+b^2}$ **Hence Proved.**

Q.4(a) If $u = \sin^{-1}(\frac{x+y}{\sqrt{x}+\sqrt{y}})$, ***Prove that***

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u.\cos 2u}{4\cos^3 u}$$ **[6]**

Ans : $u = \sin^{-1}(\frac{x+y}{\sqrt{x}+\sqrt{y}})$

Put x = xt and y = yt to find degree.

$\therefore u = \sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$

$\therefore \sin u = t^{1/2}.\frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{\frac{1}{2}}.f(x,y)$

The function sin u is homogeneous with degree ½.

But sin u is the function of u and u is the function of x and y.

By Euler's theorem ,

$$xu_x + yu_y = G(u) = n.\frac{f(u)}{f'(u)} = \frac{1}{2}\tan u$$

$$\therefore\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)[G'(u) - 1]$$

$$= \frac{1}{2}\tan u[\frac{sec^2u-2}{2}]$$

$$= \frac{1}{4}\tan u[\frac{tan^2u-1}{1}]$$

$$= \frac{1}{4} \times \frac{\sin u}{\cos u}[\frac{sin^2u-cos^2u}{cos^2u}]$$

$$\therefore\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-sinu.cos2u}{4cos^3u}$$

Hence Proved.

(b) Using encoding matrix $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ encode and decode the message "ALL IS WELL" . [6]

Ans : Let encoding matix A = $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

The message is ALL IS WELL and Let B is the matrix in number form,

$$B = \begin{bmatrix}1 & 12 & 9 & 0 & 5 & 12\\12 & 0 & 19 & 23 & 12 & 0\end{bmatrix}$$

$$A.B = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 12 & 9 & 0 & 5 & 12\\12 & 0 & 19 & 23 & 12 & 0\end{bmatrix}$$

$$C = \begin{bmatrix}13 & 12 & 28 & 23 & 17 & 12\\12 & 0 & 19 & 23 & 12 & 0\end{bmatrix}$$

The encoded message is given by,

13 12 12 0 28 19 23 23 17 12 12 0

"MLL ASWWQLL "

Inverse of encoding matix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is given by ,

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

Decoded matrix is given by ,

$$B = A^{-1}.C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

(c) Solve the following equation by Gauss Seidal method :

$$10x_1 + x_2 + x_3 = 12$$

$$2x_1 + 10x_2 + x_3 = 13$$

$$2x_1 + 2x_2 + 10x_3 = 14$$

[8]

Ans : By Gauss Seidal method ,

Given eqn : $10x_1 + x_2 + x_3 = 12$

$2x_1 + 10x_2 + x_3 = 13$

$2x_1 + 2x_2 + 10x_3 = 14$

From given eqn : $|10|>|1|+|1|$

$|10|>|2|+|1|$

$|10|>|2|+|2|$

The given eqn are in correct order.

$$\therefore\ x_1 = \frac{1}{10}[12 - x_2 - x_3]$$

$$\therefore\ x_2 = \frac{1}{10}[13 - 2x_1 - x_3]$$

$\therefore\ x_3 = \frac{1}{10}[14 - 2x_2 - 2x_1]$

I) For 1st iteration : take $x_2 = 0, x_3 = 0$

$x_1 = \frac{1}{10}[12] = 1.2$

$x_1 = 1.2\ , x_3 = 0$ gives $x_2 = 1.06$

$x_1 = 1.2, x_2 = 1.06$ gives $x_3 = 0.948$

II) For 2nd iteration : take $x_2 = 1.06, x_3 = 0.948$

$x_1 = \frac{1}{10}[12 - 1.06 - 0.948] = 0.9992$

$x_1 = 0.992, x_3 = 0.948$ gives $x_2 = 1.0068$

$x_1 = 0.992, x_2 = 1.0068$ gives $x_3 = 1.0002$

III) For 3rd iteration : $x_2 = 1.0068, x_3 = 1.0002$

$x_1 = \frac{1}{10}[12 - 1.0068 - 1.0002] = 0.9993$

$x_1 = 0.993, x_3 = 1.0002$ gives $x_2 = 1.00$

$x_1 = 0.993, x_2 = 1.00$ gives $x_3 = 1.00$

Result : $x_1 = 1.00, x_2 = 1.00, x_3 = 1.00$

Q.5(a) If u = $e^{xyz} f\left(\frac{xy}{z}\right)$ $where\ f(\frac{xy}{z})$ is an arbitrary function of $\frac{xy}{z}$.

Prove that : $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz.u$ [6]

Ans: let $\frac{xy}{z} = w$ $\therefore$ u = $e^{xyz}.f(w)$

Diff. u w.r.t. x partially,

$$\frac{\partial u}{\partial x} = e^{xyz} f'(w) + f(w).e^{xyz}.yz$$

Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial y} = e^{xyz} f'(w) + f(w).e^{xyz}.xz$$

Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial z} = e^{xyz}f'(w) + f(w).e^{xyz}.xy$$

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = xe^{xyz}f'(w) + f(w).e^{xyz}.xyz + ze^{xyz}f'(w) + f(w).e^{xyz}.xyz \quad ...(1)$$

$$y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = ye^{xyz}f'(w) + f(w).e^{xyz}.xyz + ze^{xyz}f'(w) + f(w).e^{xyz}.xyz \quad ...(2)$$

From (1) and (2),

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz.u$$

Hence Proved.

(b)Prove that $sin^5\theta = \frac{1}{16}[sin\ 5\theta - 5sin\ 3\theta + 10sin\ \theta]$ **[6]**

Ans : let x = $\cos\theta + i\sin\ \theta$ $\quad \frac{1}{x} = \cos\theta - i\sin\ \theta$

$$2\cos\theta = x + \frac{1}{x} \qquad \sin\ \theta = \frac{1}{2i}(x - \frac{1}{x})$$

For sin θ *take fifth power on both sides* ,

$$sin^5\theta = [\frac{1}{2i}(x-\frac{1}{x})]^5 = \frac{1}{32i}[x^5 - \frac{1}{x^5} - 5\left(x^3 - \frac{1}{x^3}\right) + 10(x^1 - \frac{1}{x^1})]$$

But $x^n = cos\ n\theta + isin\ n\theta \quad , \quad x^{-n} = cos\ n\theta - isin\ n\theta$

$$x^n - x^{-n} = 2isin\ n\theta$$

$$\therefore \quad sin^5\theta = \frac{1}{32i}[\ 2isin\ 5\theta - 5 \times 2isin\ 3\theta + 10 \times 2isin\ \theta\ ]$$

$$\therefore \quad sin^5\theta = \frac{1}{16}[sin\ 5\theta - 5sin\ 3\theta + 10sin\ \theta]$$

(c) i) Prove that log(sec x)=$\frac{1}{2}x^2 + \frac{1}{12}x^4 + \cdots$

ii) Expand $2x^3 + 7x^2 + x - 1$ *in powers of* $x - 2$. **[8]**

Ans : (i) Let E = log (sec x)

= - log (cos x)

= - log [1- ($\frac{x^2}{2!} - \frac{x^4}{4!}$)]

= $-[-\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) - \frac{1}{2}\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) + \cdots]$

$$E = log(sec\, x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + ..$$

(ii) let f(x)= $2x^3 + 7x^2 + x - 1$

Here a = 2

$f(x) = 2x^3 + 7x^2 + x - 1$ $\quad\quad f(2) = 45$

$f'(x) = 6x^2 + 14x + 1$ $\quad\quad f'(2) = 53$

$f''(x) = 12x + 14$ $\quad\quad f''(2) = 38$

$f'''(x) = f'''(2) = 12$

Taylor's series is :

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$2x^3 + 7x^2 + x - 1 = 45 + (x - 2)53 + \frac{(x-2)^2}{2!}38 + \frac{(x-a)^3}{3!}12$$

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3$$

Q.6(a) Prove that $\sin^{-1}(cosec\, \theta) = \frac{\pi}{2} + i.log(cot\, \frac{\theta}{2})$ **[6]**

Ans : we have to prove this $\sin^{-1}(cosec\, \theta) = \frac{\pi}{2} + i.log(cot\, \frac{\theta}{2})$

$$(cosec\, \theta) = sin[\frac{\pi}{2} + i.log\left(cot\, \frac{\theta}{2}\right)]$$

R.H.S = $sin[\frac{\pi}{2} + i.log\left(cot\, \frac{\theta}{2}\right)]$

= $cos\, [i.log\left(cot\, \frac{\theta}{2}\right)]$ ………{ sin ($\frac{\pi}{2} + x$)=cos x }

= $cos\, hlog\left(cot\, \frac{\theta}{2}\right)$ ……….{ cos ix = cos hx }

$$= \frac{1}{2}[e^{log\left(cot\frac{\theta}{2}\right)} + e^{-log\left(cot\frac{\theta}{2}\right)}]$$ ....{ cos hx = $\frac{1}{2}[e^x + e^{-x}]$ }

$$= \frac{1}{2}[cot\ \frac{\theta}{2} + \frac{1}{cot\frac{\theta}{2}}]$$

$$= \frac{1}{2} tan\ \frac{\theta}{2}[\ 1 + cot^2\frac{\theta}{2}]$$

$$= \frac{1}{2} tan\ \frac{\theta}{2}[\frac{sin^2\frac{\theta}{2}+cos^2\frac{\theta}{2}}{sin^2\frac{\theta}{2}}]$$ ....{ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ = $\frac{1}{cot\ \theta}$ }

$$= \frac{1}{2} \times \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} \times \frac{1}{sin^2\frac{\theta}{2}}$$

$$= \frac{1}{sin\ \theta}$$

$$= cosec\ \theta \quad = \quad L.H.S$$

$$\therefore (cosec\ \theta) = sin[\frac{\pi}{2} + i.log\left(cot\ \frac{\theta}{2}\right)$$

$$\therefore \sin^{-1}(cosec\ \theta) = \frac{\pi}{2} + i.log(cot\ \frac{\theta}{2})$$ **Hence Proved.**

(b) Find non singular matrices P and Q such that A= $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **[6]**

Ans : A = $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

For PAQ form ,

A = $I_{3\times3} \,.\, A_{3\times4} \,.\, I_{3\times3}$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - R_1,$

$$\begin{bmatrix}1 & 2 & 3 & 2\\ 0 & -1 & -1 & -3\\ 0 & -1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1,$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & -1 & -3\\ 0 & -1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$R_3 + R_2,$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & -1 & -3\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 1 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$C_3 - C_2, C_4 - 3C_2,$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 1 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -1 & 4\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$-R_2,$

$$\left[\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0\end{array}\right] = \begin{bmatrix}1 & 0 & 0\\ 2 & -1 & 0\\ -3 & 1 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -1 & 4\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Now A is in Normal form .

Compare this w.r.t A=PAQ form ,

$\therefore$ **P=** $\begin{bmatrix}1 & 0 & 0\\ 2 & -1 & 0\\ -3 & 1 & 1\end{bmatrix}$ **Q =** $\begin{bmatrix}1 & -2 & -1 & 4\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

$\therefore$ Rank of given matrix A is 2.

(c) Obtain the root of $x^3 - x - 1 = 0$ by Regula Falsi Method

(Take three iteration). [8]

Ans : **Equation :** $x^3 - x - 1 = 0$

$\therefore \ f(x) = x^3 - x - 1$

$f(0) = -1 < 0 \ \ and \ \ f(1) = -1 < 0 \ and \ \ f(2) = 5 > 0.$

Root of given eqn lies between 1 and 2.

(x_0, y_0) = (1,-1) (x_1, y_1) = (2,5)

$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ = 1.2249

$f(x_2) = -0.3871 < 0$

Next iteration :

(x_0, y_0) = (1.2249,-0.3871)

(x_1, y_1) = (1.667,1.9654)

$\therefore \ x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ =1.2976

$f(x_2) = -0.1127 < 0$

Next iteration :

(x_0, y_0) = (1.2976,-0.1127)

(x_1, y_2) = (1.667,1.9654)

$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = 1.3176$

The root of given eqn after 3rd iteration is 1.3176.

SEMESTER – 1
APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.
(2) Attempt any 3 questions from remaining five questions.

Q.1 (a) Show that $\sec h^{-1}(\sin \theta) = \log \cot\left(\frac{\theta}{2}\right)$. [3]

ANS: LHS = $\sec h^{-1}(\sin \theta)$

Let $y = \sec h^{-1}(\sin \theta)$

$\sec hy = \sin \theta$

$$\frac{1}{\sin\theta} = \frac{1}{\sec hy}$$

$\cos hy = \operatorname{cosec} \theta$

$y = \cos h^{-1}(\operatorname{cosec} \theta)$

but $\cos h^{-1}x = \log\left|x + \sqrt{x^2-1}\right|$

$\therefore y = \log\left|\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right|$

$\therefore y = \log\left|\operatorname{cosec}\theta + \cot\theta\right|$

$$= \log\left|\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right|$$

$$= \log\left|\frac{1+\cos\theta}{\sin\theta}\right|$$

$$= \log\left|\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right|$$

$$= \log\left|\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right|$$

$= \log \cot\left(\frac{\theta}{2}\right)$.

= RHS

$\therefore$ LHS = RHS

Hence proved

(b)Show that a matrix A = $\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$ is unitary. [3]

ANS: Given $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

To prove unitary, we have to prove $AA^{\theta} = I$

$$\therefore A^{\theta} = \frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$$

$\therefore$ LHS = AA^{θ}

$= \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 2+2+0 & -2i+2i+0 & 0+0+0 \\ 2i-2i+0 & 2+2+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

LHS= I

= RHS

LHS =RHS

Hence proved.

(c)Evaluate $\lim_{x\to 0} \sin x \log x$.   [3]

ANS: Let L = $\lim_{x\to 0} \sin x \log x$

L = $\lim_{x\to 0} \frac{\log x}{cosec\, x}$

L = $\lim_{x\to 0} \frac{\frac{1}{x}}{-cosec\, x \cot x}$ ………………………… (By L hospital)

L = $\lim_{x\to 0} \frac{-\sin x \tan x}{x}$

L = $\lim_{x\to 0} -\tan x$

L = 0

(d)Find the n^{th} derivative of $y=e^{ax}\cos^2 x \sin x$.   [3]

ANS: Given $y=e^{ax}\cos^2 x \sin x$

$y= e^{ax}\left(\frac{1+\cos 2x}{2}\right)\sin x$

$y= \frac{1}{2}(e^{ax}\sin x + e^{ax}\cos 2x \sin x)$

$y= \frac{1}{2}\left(e^{ax}\sin x + \frac{1}{2}e^{ax}(\sin 3x - \sin x)\right)$

$y= \frac{1}{2}\left(\frac{1}{2}e^{ax}\sin 3x + \frac{1}{2}e^{ax}\sin x\right)$

Diff n times,

$y_n = \frac{1}{2}\left(\frac{1}{2}e^{ax}(\sqrt{a^2+9})^n \sin(3x + n\tan^{-1}\frac{3}{a}) + \frac{1}{2}e^{ax}(\sqrt{a^2+1})^n \sin(x + n\tan^{-1}\frac{1}{a})\right)$.

(e)If x = r cos θ and y= r sin θ prove that JJ^{-1}=1.   [4]

ANS: Given x =r cos θ and y=r sin θ i.e. x,y → f (r,Θ)

$\frac{\partial x}{\partial r} = \cos\Theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$\therefore J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\Theta + \sin^2\Theta) = r.$

$\therefore J = r$ ……………………. (1)

Now, to find values of r and Θ

$\therefore x^2 + y^2 = r^2(\cos^2\Theta + \sin^2\Theta) = r^2$

$\therefore r = \sqrt{x^2 + y^2}$ and $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\Theta$

$\therefore \Theta = \tan^{-1}\frac{y}{x}$

$$\therefore J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \text{ ………………… (2)}$$

From 1 and 2, we get

Hence, $JJ' = r.\frac{1}{r} = 1$

Hence proved

(f)Using coding matrix A=$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ encode the message THE CROW FLIES AT MIDNIGHT. [4]

ANS:

T = 20 H = 8 E = 5 C = 3 R = 18 O = 15 W = 23 F = 6 L = 12 I = 9 E = 5
S = 19 A = 1 T = 20 M = 13 I = 9 D= 4 N = 14 I = 9 G = 7 H = 8 T=20

C = AB

$$B = \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$$

$$\therefore \quad C = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$$

$$C = \begin{bmatrix} 48 & 13 & 51 & 52 & 33 & 29 & 22 & 35 & 22 & 25 & 36 \\ 68 & 18 & 69 & 75 & 45 & 34 & 23 & 48 & 26 & 34 & 44 \end{bmatrix}$$

Q.2] (a) Find all values of $(1+i)^{\frac{1}{3}}$ and show that their continued product is (1 + i). [6]

ANS: Let Z = $(1+i)^{\frac{1}{3}}$

$Z = [\sqrt{2}\ (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^{\frac{1}{3}}$

$Z = (\sqrt{2})^{\frac{1}{3}} . [\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin(2k\pi + \frac{\pi}{4})]^{\frac{1}{3}}$

$Z = 2^{\frac{1}{6}}[\cos(\frac{8k\pi+\pi}{12}) + i\sin(\frac{8k\pi+\pi}{12})]$

Putting k = 0, 1, 2.

$Z_0 = 2^{\frac{1}{6}} . e^{\frac{i\pi}{12}}$

$Z_1 = 2^{\frac{1}{6}} . e^{\frac{9i\pi}{12}}$

$Z_2 = 2^{\frac{1}{6}} . e^{\frac{17i\pi}{12}}$

$\therefore Z_0\, Z_1\, Z_2 = 2^{\frac{3}{6}} . e^{\frac{27i\pi}{12}}$

$= 2^{\frac{1}{2}} . e^{\frac{9i\pi}{4}}$

$= \sqrt{2}\ (\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4})$

$= \sqrt{2}\ (\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

= (1 + i).

(b)Find the non-singular matrices P & Q such that PAQ is in normal form where $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$. [6]

ANS. Given matrix is A= $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

The order of matrix is 3 x 4

$\therefore A = I_3\ A\ I_4$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate C_2-$2C_1$; C_3-$3C_1$; C_4-$4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $\frac{C_2}{-3}$; $\frac{C_3}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -5 \\ 0 & 2 & 2 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{3}{2} & -4 \\ 0 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate R_3-$2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{3}{2} & -4 \\ 0 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Operate $\frac{R_3}{-12}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{12} & \frac{1}{6} & \frac{-1}{12} \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$[I_3, 0]$ = PAQ ie PAQ is in normal form, Where, $P=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{12} & \frac{1}{6} & \frac{-1}{12} \end{bmatrix}$ and $Q=\begin{bmatrix} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c)Find maximum and minimum values of $x^3+3xy^2-15x^2-15y^2+72x$. **[8]**

ANS: Given $f(x) = x^3+3xy^2-15x^2-15y^2+72x$… (1)

STEP 1] for maxima, minima, $\frac{\partial f}{\partial x}=0; \frac{\partial f}{\partial y}=0$

$3x^2+3y^2-30x+72=0$ and $6xy-30y=0$

$\therefore y(6x-30)=0$

$y=0, x=5$

For x=5; From Equation $3x^2+3y^2-30x+72=0$, we get $y^2-1=0$

$Y=\pm 1$

Hence (4,0) , (6,0) , (5,1) ,(5,-1) are the stationary points.

STEP 2] Now, $r=\frac{\partial^2 f}{\partial x^2}=6x-30$;

$S=\frac{d^2 f}{\partial x \partial y}=6y$;

$t=\frac{\partial^2 f}{\partial y^2}=6x-30$

STEP 3] for $(x, y) \equiv (4, 0)$, r=-6, s=0, t= -6;

$rt-s^2=(-6)(-6)-0=36>0$ and $r<0$.

This shows that the function is maximum at (4, 0)

$\therefore$From Equation (1)

$F_{max}=f(4, 0)=4^3+0-15(4^2)+0+72(4)=64-240+288$

$F_{max}=112$

STEP 4] For $(x,y)\equiv(6,0)$

r=6, s=0, t=6

$rt-s^2=36$ but $r=6>0$

This shows that function is minimum at (6, 0).

From Equation (1),

$F_{min}=f(6,0)=6^3+0-15(6)^2+0+72(6)=108$.

STEP 5] For$(x, y) \equiv (5, 1)$

r=0, s=6, t=0;

$(rt-s^2)<0$

This shows that at (5, 1) and (5,-1) function is **neither maxima nor minima**.

These points are **saddle points.**

Q.3] (a) If u=e^{xyz} f $\left(\frac{xy}{z}\right)$ prove that $x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}$=2xyzu and y$\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}$=2xyzu and hence show that $x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}$. [6]

ANS: U=e^{xyz} f $\left(\frac{xy}{z}\right)$

$$\frac{\partial u}{\partial x}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{y}{z}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xy]$$

$$x\frac{\partial u}{\partial x}=e^{xyz}\left[\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf\left(\frac{xy}{z}\right)\right]\ \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$\frac{\partial u}{\partial y}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{x}{z}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xz]$$

$$y\frac{\partial u}{\partial y}=e^{xyz}\left[\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf\left(\frac{xy}{z}\right)\right]\ \ldots\ldots\ldots\ldots\ldots\ (2)$$

$$\frac{\partial u}{\partial z}=e^{xyz}\left[f'\left(\frac{xy}{z}\right)\times\frac{xy}{z^2}\right]+f\left(\frac{xy}{z}\right)[e^{xyz}\times xy]$$

$$z\frac{\partial u}{\partial z}=e^{xyz}\left[-\frac{xy}{z}f'\left(\frac{xy}{z}\right)+xyzf\left(\frac{xy}{z}\right)\right]\ \ldots\ldots\ldots\ldots\ldots\ (3)$$

Adding 1 and 3, we get

$$x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2e^{xyz}xyzf\left(\frac{xy}{z}\right)=2xyzu$$

Adding 2 and 3, we get

$$y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2e^{xyz}xyzf\left(\frac{xy}{z}\right)=2xyzu$$

For deduction,

$$x\frac{\partial u}{\partial x}+z\frac{\partial u}{\partial z}=2xyzu$$

Diff w.r.t z

$$x\frac{\partial^2 u}{\partial x\partial z}+\left[z\frac{\partial^2 u}{\partial z}+\frac{\partial u}{\partial z}(1)\right]=2xy[z\frac{\partial u}{\partial z}+u(1)]$$

$$x\frac{\partial^2 u}{\partial x\partial z}=(2xyz-1)\frac{\partial u}{\partial z}-z\frac{\partial^2 u}{\partial z^2}+2xyu\ \ldots\ldots\ldots\ldots\ldots\ (4)$$

Similarly,

$$y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2xyzu$$

Diff w.r.t z and solving, we get

$$y\frac{\partial^2 u}{\partial y\partial z}=(2xyz-1)\frac{\partial u}{\partial z}-z\frac{\partial^2 u}{\partial z^2}+2xyu\ \ldots\ldots\ldots\ldots\ldots\ (5)$$

∴ From 4 and 5, we get

$$x\frac{\partial^2 u}{\partial z\partial x}=y\frac{\partial^2 u}{\partial z\partial y}.$$

(b)By using Regular falsi method solve 2x – 3sin x – 5 = 0. [6]

ANS: Let f(x) = 2x – 3sin x - 5

f(1) = -5

f(2) = -5.5244

f(3) = -3.7379 < 0

f(4) = 0.5766 > 0

∴Roots lies between 2 and 3

Iteration	a	b	f(a)	f(b)	x = $\frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
I.	2	3	-3.7279	0.5766	2.8660	-0.0841
II.	2.866	3	-0.0841	0.5766	2.8831	-0.0009
III.	2.8831	3	0.0009	0.5766	2.8833	-

(c)if y=sin[log(x^2+2x+1)] then prove that $(x+1)^2y_{n+2}$ +(2n +1)(x+ 1)y_{n+1} + (n^2+4)y_n=0. **[8]**

ANS: We have,

y=sin [log(x^2+2x+1)]

Diff with x

y_1 = cos [log(x^2+2x+1)] × $\frac{1}{x^2+2x+1}$ × (2x + 2)

y_1= cos [log(x^2+2x+1)] × $\frac{2(x+1)}{(x+1)^2}$

y_1= cos [log(x^2+2x+1)] × $\frac{2}{x+1}$

(x + 1) y_1=2 cos [log(x^2+2x+1)]

Diff with x again,

(x + 1) y_2 + y_1 = - 2 sin [log(x^2+2x+1)] × $\frac{1}{x^2+2x+1}$ × (2x + 2)

(x + 1) y_2 + y_1 = - 2 sin [log(x^2+2x+1)] × $\frac{2(x+1)}{(x+1)^2}$

$(x + 1)^2$ y_2 + (x + 1)y_1 = - 4 sin [log(x^2+2x+1)]

$(x + 1)^2$ y_2 + (x + 1)y_1 = - 4y

By Leibnitz Theorem,

$\left[y_{n+2}(x+1)^2 + ny_{n+1}.2(x+1) + \frac{n(n-1)}{2!}y_n.2\right]$ + $[y_{n+1}.(x+1) + ny_n(1)$] = $-4y_n$

$y_{n+2}(x+1)^2$ +(2n + 1)(x + 1) y_{n+1} + (n^2 + 4) y_n = 0

Q.4] (a) State and prove Euler's Theorem for three variables. **[6]**

ANS:

Statement: If u=f(x, y, z) is a homogeneous function of degree n, then -

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

Let, u=f(x, y, z) is a homogeneous function of degree n.

Putting X = x t, Y = y t, Z = z t.

f(X,Y,Z) = t^n f(x,y,z) ………. (1)

Diff LHS w.r.t t,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial X}.\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial Z}\frac{\partial Z}{\partial t}$$

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} \ldots\ldots (2)$$

Diff RHS w.r.t. t,

$$\frac{\partial f}{\partial t} = nt^{n-1}f(x, y, z)$$

Now put t = 1, we get $\frac{\partial f}{\partial t} = nf(x, y, z)$ ……… (3)

From equation 2 and 3, we get

$$x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} = nf(x, y, z)$$

$$x\frac{\partial f}{\partial X}. + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} = nu$$

Hence proved

(b) By using De Moirés Theorem obtain tan 5θ in terms of tan θ and show that $1 - 10\tan^2(\frac{\pi}{10}) + 5\tan^4(\frac{\pi}{10}) = 0$. [6]

ANS: $(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta + 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating both sides we get,

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\therefore \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

But $\tan 5\theta = \sin 5\theta / \cos 5\theta$

$= (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin\theta) / (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$

Dividing by $\cos^5\theta$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

for deduction, put $\theta = \frac{\pi}{10}$

$$\cot 5x\frac{\pi}{10} = \frac{1 - 10\tan^2\frac{\pi}{10} + 5\tan^4\frac{\pi}{10}}{5\tan\frac{\pi}{10} - 10\tan^3\frac{\pi}{10} + \tan^5\frac{\pi}{10}}$$

$\therefore 1 - 10\tan^2(\frac{\pi}{10}) + 5\tan^4(\frac{\pi}{10}) = 0.$

Hence proved.

(c)Investigate for what values of λ and μ the equations [8]

2x + 3y + 5z = 9

7x + 3y - 2z = 8

$2x + 3y + \lambda z = \mu$ have

A. No solutions

B. Unique solutions

C. An infinite number of solutions.

ANS: Consider the system of equation of

$2x + 3y + 5z = 9$

$7x + 3y - 2z = 8$

$2x + 3y + \lambda z = \mu$

The above system is given as Ax=B

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$$

Where $A = \begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix}$ $\quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

And $B = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$

$R_3 - R_2$

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu - 9 \end{bmatrix}$$

(A) For no solution,

$\rho(A) \neq \rho(A\ B)$

$\therefore \lambda - 5 = 0$ and $\mu - 9 \neq 0$

$\therefore \lambda = 5$ and $\mu \neq 9$

(B) For a unique solution

$\rho(A) = \rho(A\ B) = 3$

$\therefore \lambda - 5 \neq 0$ and μ may be anything

$\therefore \lambda \neq 5$ for all values of μ

(C) For infinite solutions,

$\rho(A) = \rho(A\ B) < 3$

$\therefore \lambda - 5 = 0$ and $\mu - 9 = 0$

$\therefore \lambda = 5$ and $\mu = 9$

Q.5] (a) Find n^{th} derivative of $\frac{1}{x^2+a^2}$. **[6]**

ANS: $y = \frac{1}{x^2+a^2}$.

$y = \frac{1}{(x+ai)(x-ai)}$.

Let $\frac{1}{(x+ai)(x-ai)} = \frac{A}{(x+ai)} + \frac{B}{(x-ai)}$

$1 = A(x - ai) + B(x + ai)$

Put x = ai,

1 = B(2ai)

B = $\frac{1}{2ai}$

Put x = -ai, we get

A = $-\frac{1}{2ai}$

$\therefore$ y = $\frac{\frac{1}{-2ai}}{(x+ai)} + \frac{\frac{1}{2ai}}{(x-ai)}$

$\therefore$ y = $\frac{1}{2ai}\left[\frac{1}{(x+ai)} - \frac{1}{(x-ai)}\right]$

Diff n times, we get

$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right]$.

(b) If z = f (x, y) where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$. [6]

ANS: Given: z = f (x, y) , $x = e^u + e^{-v}$ .... (1)

$y = e^{-u} - e^v$ .... (2)

By Chain Rule,

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ .... (3)

And

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ .... (4)

$\therefore$From equation 1 and 2,

$\frac{\partial x}{\partial u} = e^u$ $\frac{\partial x}{\partial v} = -e^{-v}$

$\frac{\partial y}{\partial u} = -e^{-u}$ $\frac{\partial y}{\partial v} = -e^v$

$\therefore$ From equation 3 and 4,

$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$ .... (5)

And

$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$ .... (6)

By Subtracting Equation 5 and 6,

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$

$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ .... (By using equation 1 and 2)

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Hence proved.

(c) Solve using Gauss Jacobi Iteration method [8]

2x + 12y + z – 4w = 13

$13x$ + 5y - 3z + w = 18
$2x$ + y – 3z + 9w = 31
$3x$ - 4y + 10z + w = 29

ANS:

x = $\frac{18-5y+3z-w}{13}$

y = $\frac{13-2x-z+4w}{12}$

z = $\frac{29-3x+4y-w}{10}$

w = $\frac{31-2x-y+3z}{9}$

Iteration	x	y	z	w
1	1.3846	1.0833	2.9	3.4444
2	1.3722	1.7590	2.5735	3.9831
3	0.9956	1.9679	2.7936	3.8019
4	0.9800	1.9519	3.0083	3.9357
5	1.0254	1.9812	2.9932	4.0126
6	1.0047	2.0005	2.9836	3.9942
7	0.9965	1.9987	2.9994	3.9934
8	1.0009	1.9984	3.0012	4.0007
9	1.0008	2.0000	2.9990	4.0004
10	0.9997	2.0001	2.9997	3.9995
11	0.9999	1.9999	3.0002	4.0000
12	1.0001	2	3	4.0001
13	1	2	3	4

∴ x = 1, y = 2, z = 3, w = 4.

Q.6] (a) If y = log [tan ($\frac{\pi}{4}+\frac{x}{2}$)] Prove that **[6]**

I. tan h $\frac{y}{2}$ = tan $\frac{x}{2}$

II. cos hy cos x = 1

ANS: 1] sin h$\frac{y}{2}$ = $\frac{e^{\frac{y}{2}}-e^{-\frac{y}{2}}}{2}$

cos h$\frac{y}{2}$ = $\frac{e^{\frac{y}{2}}+e^{\frac{-y}{2}}}{2}$

tan h $\frac{y}{2}$ = $\frac{\sin h\frac{y}{2}}{\cos h\frac{y}{2}}$

$$= \frac{\frac{e^{\frac{y}{2}} - e^{\frac{-y}{2}}}{2}}{\frac{e^{\frac{y}{2}} + e^{-\frac{y}{2}}}{2}}$$

$$= \frac{e^{y} - 1}{e^{y} + 1}$$

But $e^{u} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

$$\therefore \tan h\,\frac{y}{2} = \frac{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} - 1}{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + 1}$$

$$= \frac{1+\tan\frac{x}{2} - 1 + \tan\frac{x}{2}}{1+\tan\frac{x}{2} + 1 - \tan\frac{x}{2}}$$

$$= \tan\frac{x}{2}$$

$$\therefore \tan h\,\frac{y}{2} = \tan\frac{x}{2}$$

2] $y = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

$e^{y} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}$$

$$e^{y} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$e^{-y} = \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$$

$$\cos hy = \frac{e^{y} + e^{-y}}{2}$$

$$= \frac{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}{2}$$

$$= \frac{(1+\tan\frac{x}{2})^2 + (1-\tan\frac{x}{2})^2}{2(1-tan^2\frac{x}{2})}$$

$$= \frac{1+2\tan\frac{x}{2} + \tan^2\frac{x}{2} + 1 + \tan^2\frac{x}{2} - 2\tan\frac{x}{2}}{2(1-tan^2\frac{x}{2})}$$

$$= \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$\therefore$ cos hy = $\frac{1}{\cos x}$

$\therefore$ cos hy cos x = $\frac{1}{\cos x}$ .cos x

cos hy cos x = 1

Hence proved

(b) If u = $\sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ prove that

$x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144}$ [tan ²u + 13]. [6]

ANS:

Given u = $\sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$

z = sin u = $\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ is homogeneous function in x and y with degree -$\frac{1}{12}$

$\therefore$We have the result,

If z = f (u) is homogeneous function of degree x and y then

$x^2\frac{\partial^2 u}{\partial^2 x}$ + $2xy\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y}$ = g (u) [g ′(u) – 1] where g(u) = n $\frac{f(u)}{f'(u)}$.

n = -$\frac{1}{12}$, f (u) = sin u, f ′(u) = cos u

$\therefore$g (u) = -$\frac{1}{12}\frac{\sin u}{\cos u}$

$\therefore$ g (u) = -$\frac{1}{12}$ tan u

$\therefore$g′ (u) = -$\frac{1}{12}$ sec² u

By above result,

$x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y}$ = -$\frac{1}{12}$ [-$\frac{1}{12}$sec² u - 1]

= $\frac{1}{12}$ [$\frac{1}{12}$sec² u + 1] = $\frac{1}{12}$ [$\frac{1+tan^2 u}{12}$ + 1]

= $\frac{1}{144}$ tan u [tan² u + 13]

$\therefore x^2\frac{\partial^2 u}{\partial^2 x}$ + 2xy $\frac{\partial^2 u}{\partial x \partial y}$ +y² $\frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144}$ [tan ²u + 13].

Hence proved

(c) Expand 2 x^3 + 7 x^2 + x – 6 in power of (x – 2) by using Taylors Theorem. [4]

ANS: By Taylor's series,

f(x) = f(a) + (x +a)f′(a) + $\frac{(x+a)^2}{2!}$ f′′(a) + $\frac{(x+a)^3}{3!}$ f′′′(a) + ……

Here,

$f(x) = 2x^3 + 7x^2 + x - 6$

$f(2) = 2(2)^3 + 7(2)^2 + 2 - 6 = 40$

$f'(x) = 6x^2 + 14x + 1$

$f'(2) = 6(2)^2 + 14(2) + 1 = 53$

$f''(x) = 12x + 14$

$f''(2) = 12(2) + 14 = 38$

$f'''(x) = 12$

$f'''(2) = 12$

$f''''(x) = 0$.

$f(x) = f(2) + (x - 2)f'(2) + \frac{(x-2)^2}{2!} f''(a) + \frac{(x-2)^3}{3!} f'''(2) + 0$

$2x^3 + 7x^2 + x - 6 = 40 + (x - 2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12)$

$2x^3 + 7x^2 + x - 6 = 2(x - 2)^3 + 19(x - 2)^2 + 53(x - 2) + 40$

(d) Expand sec x by McLaurin's theorem considering up to x^4 term. **[4]**

ANS: Let y = sec x

$y = 1/(\cos x)$

$y = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots}$

$y = (1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots)^{-1}$

$y = 1 - (\frac{-x^2}{2} + \frac{x^4}{24}) + (\frac{-x^2}{2} + \ \ldots)^2 + \ldots$

$y = 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \ldots\ldots$

$\therefore$ **$y = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \ldots\ldots$**

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APPLIED MATHEMATICS I MAY 2019 PAPER SOLUTIONS

Q1)a) If $\mathrm{u} = \log\left(\frac{x}{y}\right) + \log\left(\frac{y}{x}\right)$**, find** $\frac{\partial u / \partial x}{\partial u / \partial y}$ **.** **(3M)**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u = 0$$

Ans :

$$\therefore \frac{\partial u / \partial x}{\partial u / \partial y} = \frac{-y}{x}$$

Q1)b) Find the value of $\tanh(\log x)$ **if** $x = \sqrt{3}$ **.** **(3M)**

Ans : Let

$$z = \tanh(\log\sqrt{3})$$
$$\therefore \tanh^{-1} z = \log\sqrt{3}$$
$$\therefore \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\log 3$$
$$\therefore \log\left(\frac{1+z}{1-z}\right) = \log\sqrt{3}$$

By componendo and dividendo

$$\frac{2}{-2z} = \frac{3+1}{1-3}$$
$$\therefore z = \frac{3-1}{3+1}$$

$$\therefore \tanh\log\sqrt{3} = \frac{3-1}{3+1} = \frac{1}{2} \quad .$$

Q1)c) Evaluate $\lim_{x\to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right]$ **.** **(3M)**

Ans: $\lim_{x\to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right]\left[\infty - \infty\right] = \lim_{x\to 3}\frac{\log(x-2)-(x-3)}{(x-3)\log(x-2)}\frac{0}{0}$

$$=\lim_{x\to 3}\frac{\frac{1}{x-2}-1}{\log(x-2)+\frac{(x-3)}{(x-2)}}=\lim_{x\to 3}\frac{-x+3}{(x-2)\log(x-2)+(x-3)}\left[\frac{0}{0}\right]$$

$$=\lim_{x\to 3}\frac{-1}{\frac{(x-2)}{(x-2)}+\log(x-2)+1}=-\frac{1}{2}\ .$$

Q1)d) If $u=r^2\cos 2\theta, v=r^2\sin 2\theta$ **, find** $\frac{\partial(u,v)}{\partial(r,\theta)}$ **.** **(3M)**

Ans: We have $\frac{\partial(x,y)}{\partial(r,\theta)}=\frac{\partial(x,y)}{\partial(u,v)}.\frac{\partial(u,v)}{\partial(r,\theta)}$. But $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} a & a \\ b & -b \end{vmatrix}=-2ab$

And $\frac{\partial(u,v)}{\partial(r,\theta)}=\begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}=4r^3$.

Q1)e) Express the matrix A = $\begin{pmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{pmatrix}$ **as the sum of a Hermitian and a Skew-Hermitian matrix.** **(4M)**

Ans: We have

$$A'=\begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix}$$

$$\therefore A^{\theta}=(\bar{A}')=\begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$$

$$\therefore A+A^{\theta}=\begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}$$

$$A-A^{\theta}=\begin{bmatrix} 6i & 2-2i & -4+3i \\ -2-2i & 0 & -1+7i \\ 4+3i & 1+7i & -2i \end{bmatrix}$$

Let $P=\frac{1}{2}(A+A^{\theta}).Q=\frac{1}{2}(\mathrm{A}-\mathrm{A}^{\theta})$.

But, we know that P is Hermitian and Q is Skew-Hermitian and A = P + Q .

$$\therefore A = P + Q = \begin{bmatrix} 2 & 1+i & (4+3i)/2 \\ 1-i & 0 & (3-3i)/2 \\ (4-3i)/2 & (3+3i)/2 & 0 \end{bmatrix} + \begin{bmatrix} 3i & 1-i & (-4+3i)/2 \\ -1-i & 0 & (-1+7i)/2 \\ (4+3i)/2 & (1+7i)/2 & -i \end{bmatrix}.$$

Q1)f) Expand $\tan^{-1} x$ **in powers of** $\left(x - \frac{\pi}{4}\right)$ **.** **(4M)**

Ans: Let

$$f(x) = \tan^{-1} x, a = \frac{\pi}{4}$$

$$\therefore f(x) = \tan^{-1} x,\ f'(x) = \frac{1}{1+x^2},\ f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$\therefore f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right),\ f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\pi/4\right)^2},\ f''\left(\frac{\pi}{4}\right) = -\frac{\pi/2}{\left[1+\left(\pi^2/16\right)\right]^2}, etc$$

$$f(x) = \mathrm{f(a)} + \mathrm{(x-a)\,f'(a)} + \frac{(x-a)^2}{2!} f''(a) + \ldots.$$

$$\therefore \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right).\frac{1}{1+\left(\pi/4\right)^2} - \left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^2 .\frac{1}{\left[1+\left(\pi^2/16\right)\right]^2} + \ldots$$

Q2)a) Expand $\sin^7\theta$ in a series of sines of multiples of θ . **(6M)**

$$x=\cos\theta+i\sin\theta$$

$$\therefore \frac{1}{x}=\cos\theta-i\sin\theta$$

$$x+\frac{1}{x}=2\cos\theta$$

$$x-\frac{1}{x}=2i\sin\theta$$

$$x^n=\cos n\theta+i\sin n\theta$$

$$\frac{1}{x^n}=\cos n\theta-i\sin n\theta$$

$$\therefore x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x^n-\frac{1}{x^n}=2i\sin n\theta$$

Now, by Binomial Theorem

$$(2i\sin\theta)^7=\left(x-\frac{1}{x}\right)^7=x^7-7x^6.\frac{1}{x}+21x^5.\frac{1}{x^2}-35x^4.\frac{1}{x^4}-21x^2.\frac{1}{x^5}+7x.\frac{1}{x^6}-\frac{1}{x^7}$$

$$\therefore (2i\sin\theta)^7=x^7-7x^5+21x^3-35x+\frac{35}{x}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}$$

$$\therefore (2i\sin\theta)^7=\left(x^7-\frac{1}{x^7}\right)-7\left(x^5-\frac{1}{x^5}\right)+21\left(x^3-\frac{1}{x^3}\right)-35\left(x-\frac{1}{x}\right)$$

$$\therefore (2i\sin\theta)^7=2i\sin7\theta-7.(2i\sin5\theta)+21.\left(2i\sin3\theta\right)-35.\left(2i\sin\theta\right)$$

$$\therefore -2^6\sin^7\theta=\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta$$

$$\therefore \sin^7\theta=\frac{-1}{2^6}\left(\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta\right)$$

Q2)b) If $y=\sin^2x\cos^3x$, then find y_n . **(6M)**

Ans: We have

$$y=\sin^2x\cos^3x$$

$$\therefore y=\sin^2x\cos^2x.\cos x=\frac{1}{4}(\sin2x)^2\cos x$$

$$\therefore y=\frac{1}{8}(1-\cos4x)\cos x=\frac{1}{8}(\cos x-\cos4x\cos x)$$

$$\therefore y=\frac{1}{8}\cos x-\frac{1}{16}\left(\cos5x+\cos3x\right)$$

By using the result $y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16}.5^n \cos\left(5x + \frac{n\pi}{2}\right) - \frac{1}{16}.3^n \cos\left(3x + \frac{n\pi}{2}\right)$$

Q2)c) Find the stationary values of $x^3 + y^3 - 3axy, a > 0$. **(8M)**

Ans: We have $f(x, y) = x^3 + y^3 - 3axy.$

Step 1:

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax$$
$$r = f_{xx} = 6x, s = f_{xy} = -3a, t = f_{yy} = 6y$$

Step 2: We now solve,

$$f_x = 0,\ f_y = 0$$
$$\therefore x^2 - ay = 0, y^2 - ax = 0$$

To eliminate y, we put $y = \frac{x^2}{a}$ in the second equation .

$$\therefore x^4 - a^3x = 0,$$
$$\therefore x(x^3 - a^3) = 0$$

Hence, x=0 or x=a

When x=0,y=0 and when x=a, y=a.

Hence, (0,0) and (a,a) are stationary points.

Step 3: (i) For x=0, y=0, $r = f_{xx} = 0, s = f_{xy} = -3a, t = f_{yy} = 0$.

Hence, $rt - s^2 = 0 - 9a < 0$.

Hence, f(x,y) is neither maximum nor minimum. It is a saddle point.

(ii) For x=a,y=a,

$$r = f_{xx} = 6a, s = f_{xy} = -3a, t = f_{yy} = 6a$$

$\therefore rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$ Hence, f(x,y) is stationary at x=a, y=a .

And $r = f_{xx} = 6a > 0, \because a > 0$

Hence f(x,y) is minimum at x=a, y=a .

Putting x=a, y=a in $x^3 + y^3 - 3axy$, the minimum value of

$f(x, y) = a^3 + a^3 - 3a^3 = -a^3$.

Q3)a) Compute the real root of $x\log_{10}^{x} - 1.2 = 0$ correct to three places of decimals using Newton-Raphson method. **(6M)**

Ans: We first note that $f(x) = x\log_{10}^{x} - 1.2$.

$$\therefore f(1) = 1\log_{10}^{1} - 1.2 = -1.2,\ f(2) = 2\log_{10}^{2} - 1.2 = -0.5979$$
$$f(3) = 3\log_{10}^{3} - 1.2 = 0.2313$$

Since f(x) changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

Now, $f'(x) = x.\dfrac{1}{x\log_e^{10}} + \log_{10}^{x} = (\log_e^{10})^{-1} + \log_{10}^{x} = 0.4343 + \log_{10}^{x}$

Hence, by Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2,3,.....$$

$$= x_n - \frac{x\log_{10}^{x} - 1.2}{0.4343 + \log_{10}^{x}}$$

For $x_0 = 3$, $x_1 = 3 - \dfrac{3\log_{10}^{3} - 1.2}{0.4343 + \log_{10}^{3}} = 2.74615$

For $x_1 = 2.74615$, $x_2 = 2.74615 - \dfrac{(2.74615).\log(2.74615) - 1.2}{0.4343 + \log 2.74615} = 2.7406$.

For x_2=2.7406, $x_3 = 2.7406$ Hence x = 2.7406 .

Q3)b) Show that the system of equations

$2x - 2y + z = \lambda x, 2x - 3y + 2z = \lambda y, -x + 2y = \lambda z$ **can possess a non-trivial solution only if $\lambda = 1, \lambda = -3$. Obtain the general solution in each case.** **(6M)**

Ans: We have $\begin{pmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The system has non-trivial solution if the rank of A is less than the number of unknowns.

The rank of A will be less than three if |A|=0.

Now, $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\therefore (2-\lambda)(\lambda^2+3\lambda-4)+2(-2\lambda+2)+1(4-3-\lambda)=0$

$\therefore (2-\lambda)(\lambda+4)(\lambda-1)-4(\lambda-1)-(\lambda-1)=0$

$\therefore (\lambda-1)[2\lambda+8-\lambda^2-4\lambda-4-1]=0$

$\therefore (\lambda-1)(-\lambda^2-2\lambda+3)=0$

$\therefore (\lambda-1)(\lambda-1)(\lambda+3)=0$

$\therefore \lambda=1, \lambda=-3$

(i) If λ=1, we have,

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By $R_2 \rightarrow R_2-2R_1$, $R_3 \rightarrow R_3+R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴x-2y+z=0. Putting $z=t_1$, $y=t_2$.

∴The solution is $x=2t_2-t_1, y=t_2, z=t_1$.

Q3)c) If $\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$ **, prove that** $\alpha=\frac{n\pi}{2}+\frac{\pi}{4}$ **and** $\beta=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ **.**

(8M)

Ans: We have

$$\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$$

$$\therefore \tan(\alpha-i\beta)=\cos\theta-i\sin\theta$$

$$\therefore \tan 2\alpha = \tan\left[(\alpha+i\beta)+(\alpha-i\beta)\right]$$

$$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\tan(\alpha-i\beta)}=\frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}$$

$$\therefore \tan 2\alpha = \frac{2\cos\theta}{0}$$

$$\therefore 2\alpha = \frac{\pi}{2}$$

$$2\alpha = n\pi + \frac{\pi}{2}$$

$$\therefore \alpha = \frac{n\pi}{2} + \frac{\pi}{4}$$

$$\tan(2i\beta) = \tan\left[(\alpha + i\beta) - (\alpha - i\beta)\right]$$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta).\tan(\alpha - i\beta)} = \frac{2i\sin\theta}{1+1} = i\sin\theta$$

$$\therefore i\tan(\mathrm{h}\,2\beta) = i\sin\theta$$

$$\therefore \tan(h2\beta) = \sin\theta$$

$$\therefore 2\beta = \tanh^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

But

$$1+\sin\theta = \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2$$

$$1-\sin\theta = \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2$$

$$\therefore 2\beta = \frac{1}{2}\log\left[\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right]^2 = \log\left[\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right]$$

$$\therefore \beta = \frac{1}{2}\log\left[\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right] = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Q4)a) Using the encoding matrix as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, encode and decode the message MOVE . **(6M)**

Ans: Step 1: To replace letters by numbers

M	O	V	E
13	15	22	5

We write this in a sequence of 2 X 2 matrix

$$\begin{bmatrix}13\\15\end{bmatrix}\begin{bmatrix}22\\5\end{bmatrix}.$$

Step 2 : To encode the message

We now premultiply each of the above column-vectors by encoding matrix $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.

$$\therefore\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}13 & 22\\15 & 5\end{bmatrix}=\begin{bmatrix}13+15 & 22+5\\0+15 & 0+5\end{bmatrix}=\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}.$$

The above message is transmitted in the following linear form taking numbers column-wise. The message is transmitted in the linear form as

28 15 27 5

Step 3: To decode the message :

The above received message is now written in a sequence of 2 X 1 column matrix as

$$\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}.$$

The above matrix is then premultiplied by the inverse of the coding matrix i.e., by

$$\therefore\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}=\begin{bmatrix}28-15 & 27-5\\0+15 & 0+5\end{bmatrix}=\begin{bmatrix}13 & 22\\15 & 5\end{bmatrix}.$$

Step 4 : To replace numbers by letters

The columns of this matrix are written in linear form as

14 15 23 27 19 20 21 4 25 27

Now it is transformed into letters using corresponding alphabets

14	15	23	27	19	20	21	4	25	27
N	O	W	*	S	T	U	D	Y	*

This is the required message.

Q4)b) If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$ **then prove that** $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ **.** **(6M)**

Ans: Let $X = e^{x-y}, Y = e^{y-z}, Z = e^{z-x}$. Then u = f(X,Y,Z)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z} e^{z-x}(-1)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Z} e^{z-x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} e^{x-y}(-1) + \frac{\partial u}{\partial Y} e^{y-z}(1) + \frac{\partial u}{\partial Z}(0)$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Y} e^{y-z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y} \mathrm{e}^{y-z}(-1) + \frac{\partial u}{\partial Z} e^{z-x}(1)$$

$$\therefore \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} e^{y-z} - \frac{\partial u}{\partial Z} e^{z-x}$$

Q4)c) If $y = a\cos(\log x) + b\sin(\log x)$ **, then show that** $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

(8M)

Ans: We have

$$y = a\cos(\log x) + b\sin(\log x)$$

$$\therefore y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$$

$$\therefore xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t x ,

$$\therefore xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x}$$

$$\therefore x^2 y_2 + xy_1 + y = 0 \quad .$$

Applying Leibnitz's theorem to each term, we get

$$x^2 y_{n+2} + n(2x) y_{n+1} + \frac{n(n-1)}{2!}(2) y_n + [x y_{n+1} + n(1) y_n] + y_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 - n + n + 1)\, y_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 + 1) y_n = 0$$

Q5)a) If 1, $\alpha, \alpha^2, \alpha^3, \alpha^4$ are the roots of $x^5 - 1 = 0$, find them and show that

(1-α)(1-α^2)(1-α^3)(1-α^4)=5. **(6M)**

Ans: We have

$$x^5 = 1 = \cos 0 + i \sin 0$$

$$\therefore x^5 = \cos(2k\pi + i \sin(2k\pi)$$

$$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + \sin\frac{2k\pi}{5}$$

Putting k = 0, 1, 2, 3, 4, we get the five roots as

$x_0 = \cos 0 + i \sin 0$

$$\therefore x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ = we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$.

Therefore, the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore \frac{x^5 - 1}{x - 1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting x = 1, we get

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

Q5)b) If $\theta = t^n e^{\frac{-r^2}{4t}}$ **,**

Find n which will make $\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right)$ **.** **(6M)**

Ans:

$$\frac{\partial \theta}{\partial t} = nt^{n-1}.e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}}.\left(\frac{-r^2}{4}\right)\left(\frac{-1}{t^2}\right)$$

$$= \frac{n}{t}.t^n.\frac{\theta}{t^n} + t^n.\frac{\theta}{t^n}\left(\frac{r^2}{4t^2}\right)$$

$$= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta$$

$$\left(\because e^{\frac{-r^2}{4t}} = \frac{\theta}{t^n}\right)$$

Also ,

$$\frac{\partial \theta}{\partial r} = t^n.e^{\frac{-r^2}{4t}}.\left(\frac{-2r}{4t}\right) = \frac{-r\theta}{2t} \quad .$$

$$\therefore r^2 \frac{\partial r}{\partial \theta} = \frac{-r^3\theta}{2t}$$

$$\therefore \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{-r^3\theta}{2t}\right) = \frac{-1}{2t}.\frac{\partial}{\partial r}(r^3\theta) = \frac{-1}{2t}\left[r^3\frac{\partial \theta}{\partial r} + 3r^2\theta\right]$$

$$= \frac{-1}{2t}\left[\frac{-r^4\theta}{2t} + 3r^2\theta\right] = r^2\left(\frac{r^2}{4t^2} - \frac{3}{2t}\right)\theta$$

$$\therefore \frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \left(\frac{r^2}{4t^2} - \frac{3}{2t}\right)\theta$$

Now equating we get

$$\frac{n}{t} = \frac{-3}{2t}$$

$$\therefore n = \frac{-3}{2}$$

Q5)c) Find the root (correct to three places of decimals) of $x^3 - 4x + 9 = 0$ lying between 2 and 3 by using Regula-Falsi method . (8M)

Ans: Let $y = f(x) = x^3 - 4x - 9$. Here, $x_1 = 2$ and $x_2 = 3$.

$$\therefore y_1 = f(x_1) = f(2) = 2^3 - 4(2) - 9 = 8 - 8 - 9 = -9 < 0$$
$$y_2 = f(x_2) = f(3) = 3^3 - 4(3) - 9 = 27 - 12 - 9 = 6 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2 to 3 , there is a root between 2 and 3 .

The root is given by

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(6) - 3(-9)}{6 - (-9)} = \frac{39}{15} = 2.6 \ .$$

Now, $y_3 = f(x_3) = f(2.6) = (2.6)^3 - 4(2.6) - 9 = -1.82 > 0$

Since f(x) changes its sign from negative to positive as x goes from 2.6 to 3 , there is a root between 2.6 and 3 .

First Iteration : Let

$$x_1 = 2.6, x_2 = 3, y_1 = -1.82, y_2 = 6$$
$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.6(6) - 3(-1.82)}{6 - (-1.82)} = 2.693$$
$$y_3 = f(x_3) = (2.693)^3 - 4(2.693) - 9 = -0.242 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2.693 to 3 , there is a root between 2.693 to 3 .

Second Iteration : Let

$$x_1 = 2.693, x_2 = 3, y_1 = -0.262, y_2 = 6$$
$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.693(6) - 3(-0.262)}{6 - (-0.262)} = 2.7058 = 2.706$$
$$y_3 = f(x_3) = (2.706)^3 - 4(2.706) - 9 = -0.009 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2.706 to 3 , there is a root between 2.706 to 3 .

Third Iteration : Let

$$x_1 = 3, x_2 = 2.706, y_1 = 6, y_2 = -0.009$$

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(0.03) - 2.706(6)}{-0.009 - 6} = 2.706$$

Hence, the root correct to three places of decimals = 2.706 .

Q6)a) Find non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **is reduced to normal form. Also find its rank.** **(6M)**

Ans: We first write A = I_3 A I_4

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

By $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ \quad .$$

Hence , the rank of A is 2 .

Q6)b) Find the principle value of $(1+i)^{1-i}$ **.** **(6M)**

Ans: Let

$$z = (1+i)^{1-i}, \therefore \log z = (1-i)\log(1+i)$$

$$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + i\tan^{-1}1\right]$$

$$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right] = \frac{1}{2}\log 2 + \frac{i\pi}{4} - \frac{i}{2}\log 2 + \frac{\pi}{4}$$

$$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = x + iy$$

$$\therefore z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$$

$$= e^{\left(\frac{1}{2}\right)\log 2 + \left(\frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

$$= \sqrt{2}e^{\frac{\pi}{4}}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

Q6)C) Solve the following equations by Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

(Take three iterations) **(8M)**

Ans: We first write the three equations as

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

(i) First Iteration : We start with the approximation y=0, z=0 and we get

$$\therefore x_1 = \frac{85}{27} = 3.15 \ .$$

We use this approximation to find y_1 i.e. we put x = 3.15 , z=0 in the second equation

$$\therefore y_1 = \frac{1}{15}[72 - 6(3.15)] = 3.54 \ .$$

We use these values of x_1 and y_1 to find z_1 i.e. we put x=3.15 and y=3.54 in the third equation

$$\therefore z_1 = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91 \ .$$

(ii) Second Iteration : We use the latest values of y and z to find x , i.e. we put y= 3.54 , z=1.91 in equation 1 , we get

$$x_2 = \frac{1}{27}[85 - 6(3.54) + 1.91] = 2.43$$

We put x = 2.43 , z = 1.91 to find y from equation 2. Thus ,

$$y_2 = \frac{1}{15}[72 - 6(2.43) - 2(1.91)] = 3.57$$

We put x = 2.43 , y = 3.57 in equation 3 to find z. Thus ,

$$z_2 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93$$

(iii) Third iteration: Putting y = 3.57 , z = 1.93 in equation (1) we get

$$x_3 = \frac{1}{27}[85 - 6(3.57) + 1.93] = 2.43$$

Putting x = 2.43 , z = 1.93 in equation 2 we get

$$y_3 = \frac{1}{15}[72 - 6(2.43) - 2(1.93)] = 3.57$$

Putting x=2.43 , y=3.57 in equation 3 we get

$$z_3 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93 \ .$$

Since the second and third iteration give the same values

x = 2.43 , y = 3.57 , z = 1.93

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APPLIED MATHS I DEC 2019 PAPER SOLUTIONS

Q1)a) If $\sin(\theta+i\varphi)=\tan\alpha+i\sec\alpha$ **, then show that** $\cos 2\theta.\cosh 2\varphi=3$ **.** **(5M)**

Ans : We have $\sin(\theta+i\varphi)=\tan\alpha+i\sec\alpha$

$$\therefore \sin\alpha\cos i\varphi+\cos\theta\sin i\varphi=\tan\alpha+i\sec\alpha$$

$$\therefore \sin\alpha\cosh\varphi+i\cos\theta\sinh\varphi=\tan\alpha+i\sec\alpha$$

Equating real and imaginary parts ,

$$\tan\alpha=\sin\theta\cosh\varphi$$
$$\sec\alpha=\cos\theta\sinh\varphi$$

But

$$\sec^2\alpha-\tan^2\alpha=1$$
$$\therefore \cos^2\theta\sinh^2\varphi-\sin^2\theta\cosh\varphi=1$$
$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\varphi-1}{2}\right)-\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\varphi}{2}\right)=1$$
$$\therefore \cosh 2\varphi-1+\cos 2\theta\cosh 2\varphi-\cos 2\theta-1-\cosh 2\varphi+\cos 2\theta+\cos 2\theta\cosh 2\varphi=4$$
$$\therefore 2\cos 2\theta\cosh 2\varphi=6$$
$$\therefore \cos 2\theta\cosh 2\varphi=3$$

Q1)b) If $u=\log(\tan x+\tan y)$ **, then show that** $\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2$ **.** **(5M)**

Ans : We have

$$\frac{\partial u}{\partial x}=\frac{1}{(\tan x+\tan y)}\sec^2 x$$

$$\therefore \sin 2x\frac{\partial u}{\partial x}=2\sin x\cos x\frac{1}{(\tan x+\tan y)}\sec^2 x=2.\frac{\tan x}{\tan x+\tan y}$$

Similarly , $\sin 2y\frac{\partial u}{\partial y}=2.\frac{\tan y}{(\tan x+\tan y)}$

$$\therefore \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2\frac{\tan x+\tan y}{(\tan x+\tan y)}=2 .$$

Similarly , prove that

$$\therefore \sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}+\sin 2z\frac{\partial u}{\partial z}=2$$

Q1)c) Express the matrix $A=\begin{bmatrix}0 & 5 & -3\\ 1 & 1 & 1\\ 4 & 5 & 9\end{bmatrix}$ **as the sum of a symmetric and skew symmetric matrix.**

(5M)

Ans : We have

$$A'=\begin{bmatrix}0 & 1 & 4\\ 5 & 1 & 5\\ -3 & 1 & 9\end{bmatrix}$$

$$\therefore A+A'=\begin{bmatrix}0 & 6 & 1\\ 6 & 2 & 6\\ 1 & 6 & 18\end{bmatrix}$$

$$A-A'=\begin{bmatrix}0 & 4 & -7\\ -4 & 0 & -4\\ 7 & 4 & 0\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A')$ and $Q=\frac{1}{2}(A-A')$.

But we know that P is symmetric and Q is skew-symmetric and A=P + Q .

$$\therefore A=P+Q=\begin{bmatrix}0 & 3 & 1/2\\ 3 & 1 & 3\\ 1/2 & 3 & 9\end{bmatrix}+\begin{bmatrix}0 & 2 & -7/2\\ -2 & 0 & -2\\ 7/2 & 2 & 0\end{bmatrix}$$

The first matrix is symmetric and the second is skew-symmetric .

Q1)d) Expand $\sqrt{1+\sin x}$ in ascending powers of x upto x^4 terms . **(5M)**

Ans : We have

$$\sqrt{1+\sin x} = \sqrt{\sin^2(x/2)+\cos^2(x/2)+2\sin(x/2)\cos(x/2)}$$

$$= \sqrt{\left[\sin(x/2)+\cos(x/2)\right]^2} = \sin(x/2)+\cos(x/2)$$

$$= \left(\frac{x}{2}\right) - \frac{1}{6}\left(\frac{x}{2}\right)^3 + \ldots\ldots\ldots + 1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{24}\left(\frac{x}{4}\right)^4 - \ldots.$$

$$= \frac{x}{2} - \frac{x^3}{48} + \ldots\ldots. + 1 - \frac{x^2}{8} + \frac{x^4}{384} - \ldots\ldots.$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \ldots\ldots.$$

Q2)a) Find non-singular matrices P and Q such that PAQ is in normal form where,

$A = \begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$**. Also find the rank of A .** **(6M)**

Ans : We first write

$$A = I_3 A I_4$$

$$\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to \dfrac{R_1}{4}$,

$$\begin{bmatrix} 1 & 3 & 1 & 6 \\ 0.5 & 4 & 2 & 2 \\ 3 & 14 & 5 & 16 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 1 & 6 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1, C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\frac{-C_2}{3}, \frac{-C_3}{2}, -\frac{C_5}{5}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 3/2 & 4/5 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}.$$

$C_3 - C_2, C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 5/6 & -7/24 \\ 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & -5/24 \\ 0 & 0 & 0 & -1/12 \end{bmatrix}.$$

C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}.$$

Q2)b) If $z = f(x, y)$ **and** $x = u\cosh v, y = u\sinh v$,prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2.$$ **(6M)**

Ans : We have ,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v \quad \text{-------- (1)}$$

And , $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}\text{usinh}\,v+\frac{\partial z}{\partial y}\text{ucosh}\,v$

$\frac{1}{u}.\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\sinh v+\frac{\partial z}{\partial y}\cosh v$ --------- (2) .

Now squaring (1) and (2) and subtracting , we get

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}.\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v+2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\text{coshv}.\sinh v-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v-\left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v-2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\text{coshv}.\sinh v$$

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 .$$

Q2)c) Prove that $\log\left(\frac{(a-b)+\text{i}(\text{a}+\text{b})}{(a+b)+i(a-b)}\right)=i\left(2n\pi+\tan^{-1}\frac{2ab}{a^2-b^2}\right)$ **.Hence evaluate** $\log\left(\frac{1+5i}{5+i}\right)$**.**

(6M)

Ans : Let a + b=A , a – b =B .

$$\therefore \log\left[\frac{B+iA}{A+iB}\right]=2n\pi i+\log\left[\frac{B+iA}{A+iB}\right]$$
$$=2n\pi i+\log(B+iA)-\log(A+iB)$$
$$=2n\pi i+\left[\log\sqrt{B^2+A^2}+i\tan^{-1}\frac{A}{B}\right]-\left[\log\sqrt{A^2+B^2}+i\tan^{-1}\frac{B}{A}\right] .$$
$$=2n\pi i+i\left[\tan^{-1}\frac{A}{B}-\tan^{-1}\frac{B}{A}\right]$$

But $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$.

$$\therefore \log\left[\frac{B+iA}{A+iB}\right]=2n\pi i+i\tan^{-1}\left[\frac{(A/B)-(B/A)}{1+(A/B)*(B/A)}\right] .$$
$$=2n\pi i+i\tan^{-1}\left(\frac{A^2-B^2}{2AB}\right)$$

But $A^2-B^2=(\text{a}+\text{b})^2-(\text{a}-\text{b})^2=4ab$ and $AB=(a+b)(a-b)=a^2-b^2$.

$$\therefore \log\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + i\tan^{-1}\frac{4ab}{2(a^2-b^2)} = i\left[2n\pi + \tan^{-1}\frac{2ab}{(a^2-b^2)}\right]$$

Q3)a) If α and β are the roots of the equation $z^2\sin^2\theta - z\sin 2\theta + 1 = 0$**, then prove that** $\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$ and $\alpha^n\beta^n = \operatorname{cosec}^{2n}\theta$. **(6M)**

Ans : Solving the quadratic equation in z ,

$$z = \frac{\sin 2\theta \pm \sqrt{\sin^2 2\theta - 4\sin^2\theta}}{2\sin^2\theta} = \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$\therefore z = \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{\sin\theta} = \frac{\cos\theta \pm \sqrt{-\sin^2\theta}}{\sin\theta}$$

$$= \frac{\cos\theta \pm i\sin\theta}{\sin\theta} = (\cos\theta \pm i\sin\theta)\operatorname{cosec}\theta$$

Let $\alpha = (\cos\theta + i\sin\theta)\operatorname{cosec}\theta$, $\beta = (\cos\theta - i\sin\theta)\operatorname{cosec}\theta$.

$$\therefore \alpha^n = (\cos\theta + i\sin\theta)^n \operatorname{cosec}^n\theta = (\cos n\theta + i\sin(n\theta))\operatorname{cosec}^n\theta$$

$$\beta^n = (\cos\theta - i\sin\theta)^n \operatorname{cosec}^n\theta = (\cos n\theta - i\sin(n\theta))\operatorname{cosec}^n\theta$$

$$\therefore \alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$$

$$\alpha^n.\beta^n = (\cos\theta + i\sin\theta)^n \operatorname{cosec}^n\theta.(\cos\theta - i\sin\theta)^n \operatorname{cosec}^n\theta$$

$$= (\cos^2 n\theta + \sin^2 n\theta)\operatorname{cosec}^{2n}\theta = \operatorname{cosec}^{2n}\theta$$

Q3)b) Solve the following equations by Gauss-Siedal Method ;

15x+2y+z = 18 , 2x+20y-3z = 19 , 3x-6y+25z = 22 .Take three iterations . **(6M)**

Ans : We first write the equations as

$$x = \frac{1}{15}[18 - 2y - z] \quad \text{-------------------(1)}$$

$$y = \frac{1}{20}[19 - 2x + 3z] \quad \text{------------------(2)}$$

$$z = \frac{1}{25}[22 - 3x + 6y] \quad \text{-------------------(3)}$$

(i) First Iteration :We start with the approximation $y_0 = 0, z_0 = 0$ and then from (1) , we get

$$x_1 = \frac{18}{15} = 1.2 \quad .$$

We use this approximation to find y from (2), i.e we put x=1.2 and z=0 in (2) and get

$$y_1 = \frac{1}{20}\left[19 - 2(1.2) + 3(0)\right] = \frac{16.6}{20} = 0.83 \quad .$$

We use these values of x and y to find z from (3) i.e. we put x=1.2 , y=0.83 in (3) and get

$$z = \frac{1}{25}\left[22 - 3(1.2) + 6(0.83)\right] = \frac{23.38}{25} = 0.9352 \quad .$$

(ii) Second iteration : We use the latest values of y and z in (1) to find x , i.e we put y=0.83 and z=0.9352 in (1) and get

$$x_2 = \frac{1}{15}\left[18 - 2(0.83) - 0.9352\right] = \frac{15.4048}{15} = 1.0270 \quad .$$

We now put x=1.027 and z=0.9352 in (2) and get

$$y_2 = \frac{1}{20}\left[19 - 2(1.027) + 3(0.9352)\right] = \frac{19.7516}{20} = 0.9876 \quad .$$

We use these values of x and y to find z from (3)

$$z_2 = \frac{1}{25}\left[22 - 3(1.0270) + 6(0.9876)\right] = \frac{28.8446}{25} = 0.9938 \quad .$$

(iii) Third Iteration : We use the latest values of y and z to find x , i.e. ,we put 0.9876 and z=0.9938 in (1) and get

$$x_3 = \frac{1}{15}\left[18 - 2(0.9876) - 0.9938\right] = \frac{15.031}{15} = 1.0021 \quad .$$

We now put x=1.0021 and z=0.9938 in (2) and get

$$y_3 = \frac{1}{20}\left[19 - 2(1.0021) + 3(0.9938)\right] = \frac{19.9772}{20} = 0.9989 \quad .$$

We use these values of x and y to find z from (3)

$$z_3 = \frac{1}{25}\left[22 - 3(1.0021) + 6(0.9989)\right] = \frac{24.9871}{25} = 0.9995 \quad .$$

Hence , we get x =1.0021 , y =0.9989 , z =0.9995 .

Q3)c) Prove that if z is a homogeneous function of two variables x and y of degree n , then $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$ **.Hence find the value of** $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$ **at x=1 , y=1 when** $z=x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+\frac{x^4+y^4}{x^2+y^2}$ **.** **(8M)**

Ans : Since z is a homogeneous function of degree n in x and y , by Euler's Theorem ,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad \text{-------------- (1)}$$

Differentiating (1) partially w.r.t x ,

$$\left(x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.1\right)+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$

Differentiating (1) partially w.r.t y ,

$$x\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$

Multiplying (2) by x and (3) by y and adding , we get ,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]=(n-1)nz$$

Further, if u is a homogeneous function of three variables x , y , z of degree n then we can prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xy\frac{\partial^2 u}{\partial y\partial z}+2xy\frac{\partial^2 u}{\partial z\partial x}=n(n-1)\,\mathrm{u}$$

For $z=x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+\frac{x^4+y^4}{x^2+y^2}$.

Putting X=xt , Y=yt , we get

F(X,Y) = $X^6\tan^{-1}\left(\frac{X^2+Y^2}{X^2+XY}\right)+\frac{X^4+Y^4}{X^2+Y^2}$.

$$\therefore f(X,Y) = x^6 t^6 \tan^{-1}\left(\frac{x^2t^2+y^2t^2}{x^2t^2+xtyt}\right)+\frac{x^4t^4+y^4t^4}{x^2t^2+y^2t^2} = x^6 t^6 \tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+t^2\frac{x^4+y^4}{x^2+y^2}$$.

Now, let $x^6 t^6 \tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)$ = u , and $t^2\frac{x^4+y^4}{x^2+y^2}$ = v .

u and v are homogeneous functions of degree 6 and 2 respectively .

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=6u \ , \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)nu=30u$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=2v$$

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=(n-1)nv=2v$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=(n-1)nu+n(n-1)v=30u+2v$$.

$$\therefore x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)nz=30x^6 t^6 \tan^{-1}\left(\frac{x^2+y^2}{x^2+xy}\right)+2t^2\frac{x^4+y^4}{x^2+y^2}$$.

Q4)a) If $\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$ **then prove that** $\alpha=\frac{n\pi}{2}+\frac{\pi}{4}, \beta=\frac{1}{2}\log\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$**.** **(6M)**

Ans : We have **,** $\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$ $\therefore \tan(\alpha-i\beta)=\cos\theta-i\sin\theta$

$$\therefore \tan 2\alpha = [\tan((\alpha+i\beta)+(\alpha-i\beta))]$$

$$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}=\frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}$$

$$\tan 2\alpha=\frac{2\cos\theta}{0}$$

$$\because 2\alpha = n\pi+\frac{\pi}{2}, \alpha=\frac{n\pi}{2}+\frac{\pi}{4}$$

Also , $\therefore \tan 2\beta = [\tan((\alpha+i\beta)-(\alpha-i\beta))]$.

$$=\frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}=\frac{2i\sin\theta}{1+1}=i\sin\theta$$

$$\text{i}\tanh 2\beta = i\sin\theta$$

$$\therefore \tanh 2\beta=\sin\theta$$

$$\therefore 2\beta = \tanh^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

But $1+\sin\theta = \left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2$

But $1-\sin\theta = \left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\right)-2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2$

$$\therefore 2\beta = \frac{1}{2}\log\left(\frac{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}\right)^2 = \log\left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]$$

$$\beta = \frac{1}{2}\log\left[\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right] = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Q4)b) Expand $x^5+x^3-x^2+x-1$ **in powers of (x-1) and hence find the value of** **(6M)**

(1) $f\left(\frac{9}{10}\right)$

(2) $f(1.01)$

Ans : Let $f(x)=x^5+x^4-x^2+x-1$ and a=1 , $\therefore f(1)=1$

$\therefore f'(x)=5x^4-4x^3+3x^2-2x+1,$ $\quad \therefore f'(1)=3$

$\therefore f''(x)=20x^3-12x^2+6x-2,$ $\quad \therefore f''(1)=12$

$\therefore f'''(x)=60x^2-24x+6,$ $\quad \therefore f'''(1)=42$

$\therefore f^{iv}(x)=120x-24,$ $\quad \therefore f^{iv}(1)=96$

$\therefore f^{v}(x)=120,$ $\quad \therefore f^{v}(1)=120$

Now, $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots$

$$\therefore f(x)=1+(x-1).3+\frac{(x-1)^2}{2!}.12+\frac{(x-1)^3}{3!}.42+\frac{(x-1)^4}{4!}.96+\frac{(x-1)^5}{5!}.120+\ldots\ldots$$

$\therefore f(x)=1+(x-1).3+6(x-1)^2+7(x-1)^3+4(x-1)^4+(x-1)^5$.

(i) To find $f\left(\frac{9}{10}\right)$, we put x= 0.9 , and x-1 = -0.1

$\therefore f(0.9)=1+3(-0.1)+6(-0.1)^2+7(-0.1)^3+4(-0.1)^4+(-0.1)^5$

$=1-0.3+0.06-0.007+0.0004-0.00005$.

$=0.7534$

(ii) To find $f(1.01)$, we put x=1.01 and (x-1) =0.01 .

$\therefore f(1.01)=1+3(0.01)+6(0.01)^2+7(0.01)^3+4(0.01)^4+(0.01)^5$

$=1+0.03+0.0006-0.000007+0.00000004-0.0000000005$.

$=01.0306$

Q4)c) For what values of λ and μ , the equations , x + y + z =6 ; x + 2y +3z =10 ; x + 2y + λz =μ ;

(i) Have a unique solution

(ii) Have infinite solution

Find the solution in each case for a possible value of μ and λ . **(8M)**

Ans :

We have $\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$.

By $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$,

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&\lambda-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\4\\\mu-10\end{bmatrix}$$.

(i) The system has unique solution if the coefficient matrix is non-singular (or the rank A , r = the number of unknowns , n=3)

This requires $\lambda-3\neq 0, \ \therefore \lambda\neq 3$.

$\therefore \lambda \neq 3$ then (μ may have any value)the system has unique solution .

(ii) If λ = 3 , the coefficient matrix and the augmented matrix becomes

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&0\end{bmatrix} \text{ and } \begin{bmatrix}1&1&1&6\\0&1&2&4\\0&0&0&\mu-10\end{bmatrix}$$.

The rank of A = 2 , and the rank of [A,B] will be also 2 if μ = 10 .

Thus if λ = 3 and μ = 10 , the system is consistent .But the rank of A(=2) is less than the number of unknowns (=3) . Hence the equations will possess infinite solutions .

Q5)a) Find the nth derivative of $y=\dfrac{1}{x^2+a^2}$. **(6M)**

Ans : We have

$$y=\frac{1}{x^2+a^2}=\frac{1}{x^2-a^2i^2}=\frac{1}{2ai}\left[\frac{1}{x-ai}-\frac{1}{x+ai}\right]$$

$$y_n=\frac{1}{2ai}\left[\frac{(-1)^n.n!}{(x-ai)^{n+1}}-\frac{(-1)^n.n!}{(x+ai)^{n+1}}\right]$$.

$$=\frac{(-1)^n.n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right]$$

Let $x=r\cos\theta, a=r\sin\theta$, so that $r^2=x^2+a^2, \theta=\tan^{-1}(\text{a/ x})$.

Now,

$$\frac{1}{(x-ai)^{n+1}}=\frac{1}{r^{n+1}(\cos\theta-i\sin\theta)^{n+1}}=\frac{1}{r^{n+1}}.\frac{1}{\cos(n+1)\theta-i\sin(n+1)\theta}$$

$$\frac{1}{(x-ai)^{n+1}}=\frac{1}{r^{n+1}}[\cos(n+1)\theta+i\sin(n+1)\theta]$$.

$$\frac{1}{(x+ai)^{n+1}}=\frac{1}{r^{n+1}(\cos\theta+i\sin\theta)^{n+1}}=\frac{1}{r^{n+1}}.\frac{1}{\cos(n+1)\theta+i\sin(n+1)\theta}$$

$$\frac{1}{(x-ai)^{n+1}}=\frac{1}{r^{n+1}}[\cos(n+1)\theta-i\sin(n+1)\theta]$$

$$\therefore \frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}=\frac{1}{r^{n+1}}.2i\sin(n+1)\theta$$.

Putting these values in $\dfrac{(-1)^n.n!}{2ai}\left[\dfrac{1}{(x-ai)^{n+1}}-\dfrac{1}{(x+ai)^{n+1}}\right]$, we get

$$y_n = (-1)^n . n! . \frac{1}{a} . \frac{1}{r^{n+1}} \sin(n+1)\theta$$.

But r = $\frac{a}{\sin\theta}$. $\left(\because a = r\sin\theta, \therefore r^{n+1} = \frac{a^{n+1}}{\sin^{n+1}\theta} \right)$.

$$y_n = (-1)^n . n! \frac{1}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$$.

Q5)b) Discuss the maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$. **(6M)**

Ans : We have f(x , y) = $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 16$.

Step I :

$$f_x = 3x^2 + y^2 - 24x + 21$$
$$f_y = 2xy - 4y$$
$$f_{xx} = 6x - 24,\ f_{xy} = 2y,\ f_{yy} = 2x - 4$$

Step II :

We now solve the equations $f_x = 0$, $f_y = 0$.

$\therefore 3x^2 + y^2 - 24x + 21 = 0$ and $2xy - 4y = 0$.

The second equation gives 2y(x-2) =0 .

$\therefore$ x = 2 or y = 0 .

When x=2 , the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$\therefore 12 + y^2 - 48 + 21 = 0$ hence , y^2 -15 = 0 , y^2 =15 , y = $\pm\sqrt{15}$.

$\therefore$ The stationary values are (2, $\sqrt{15}$) , (2, - $\sqrt{15}$) .

When y = 0 , the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$3x^2 - 24x + 21 = 0$, $x^2 - 8x + 7 = 0$.

(x-7) (x-1) = 0 , hence x = 1, 7 .

Therefore , the stationary values are (1,0) , (7,0) .

Step III :

(i) For x =2 , y = $\sqrt{15}$,

$r = f_{xx} = 12 - 24 = -12, \text{s} = f_{xy} = 2\sqrt{15}, \text{t} = f_{yy} = 4 - 4 = 0$.

$\therefore rt - s^2 = 0 - 60 = -60 < 0$.

$\therefore$ f(x,y) is neither maximum nor minimum. It is a saddle point .

(ii) For x =2 , y =- $\sqrt{15}$,

$r = f_{xx} = 12 - 24 = -12, \text{s} = f_{xy} = -2\sqrt{15}, \text{t} = f_{yy} = 4 - 4 = 0$.

$\therefore rt - s^2 = 0 - 60 = -60 < 0$.

$\therefore$ f(x,y) is neither maximum nor minimum. It is a saddle point .

(iii) For x =1 , y =0 ,

$r = f_{xx} = 6 - 24 = -18, \text{s} = f_{xy} = 0, \text{t} = f_{yy} = 2 - 4 = -2$

$\therefore rt - s^2 = 36 - 0 = 36 > 0$ and r = -18 , negative

$\therefore$ (1,0) is a maxima .

$\therefore$ The maximum value = 1 + 0 – 12 – 0 + 21 = 20 .

(iv) For x =7 , y =0 ,

$r = f_{xx} = 42 - 24 = 18, \text{s} = f_{xy} = 0, \text{t} = f_{yy} = 14 - 4 = 10$.

$\therefore rt - s^2 = 180 - 0 = 180 > 0$

Hence , (7,0) is a minima .

The minimum value = 343 + 0 -588- 0 +147 +10 =-88 .

Q5)c) Prove that if A and B are two unitary matrices then AB is also unitary. Verify the result when

$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ **and** $B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ **.** **(8M)**

Ans : We have

$$(AB)(AB)^\theta = (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta$$.

$$= AIA^\theta \qquad [\ \because \text{B is unitary}\ ]$$.

$$= AA^\theta = I \qquad [\ \because A \text{ is unitary}\ ]$$.

Similarly , we can prove that $(AB)^\theta (AB) = I$.

Hence , AB is also unitary .

Now, $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

$$\therefore A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now, $B = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$$\therefore B' = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \quad , \quad \therefore B^{\theta} = (\bar{B}') = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$B^{\theta}B = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$B^{\theta}B = \frac{1}{4}\begin{bmatrix} (1-i^2)+(1-i^2) & -(1-i)^2+(1-i)^2 \\ -(1+i)^2+(1+i)^2 & (1-i)^2+(1-i)^2 \end{bmatrix}$$

$$B^{\theta}B = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence it is proved that if A and B are two unitary matrices then AB is also unitary and the result is verified .

Q6)a) If $x=\cosh\left(\frac{1}{m}\log y\right)$ **, prove that** $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ **.** **(6M)**

Ans : $$x=\cosh\left(\frac{1}{m}\log y\right) .$$

$$\cosh^{-1}x=\left(\frac{1}{m}\log y\right)$$

$$\log y=m\log\left(x+\sqrt{x^2-1}\right)=\log\left(x+\sqrt{x^2-1}\right)^m$$

Differentiating w.r.t x ,

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{2x}{2\sqrt{x^2-1}}\right) .$$

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$=m\frac{\left(\sqrt{x^2-1}+x\right)^m}{\sqrt{x^2-1}}=\frac{my}{\sqrt{x^2-1}}$$

$$y_1\sqrt{x^2-1}=my$$
$$(x^2-1)y=m^2y^2$$

Differentiating w.r.t x ,

$$y_1\sqrt{x^2-1}=my$$
$$(x^2-1)2y_1y_2+2xy_1^2=m^2 2yy_1$$
$$(x^2-1)y_2+xy_1=m^2y$$

Differentiating n times using Leibnitz Theorem ,

$$(x^2-1)y_{n+2}+n.2xy_{n+1}+\frac{n(n-1)}{2!}2y_n+xy_{n+1}+ny_n=m^2y_n .$$

$$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

Q6)b) Find a root of the equation $xe^x = \cos x$ using the Regula Falsi Method correct to three decimal places . (6M)

Ans :

$$f(x) = \cos x - xe^x = 0$$
$$f(0) = 1$$
$$f(1) = \cos 1 - e = -2.17798$$

The root lies between 0 and 1 .

Taking $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{-1}{-2.17798 - 1} = \frac{1}{3.17798} = 0.3147 .$$

Now, $\cos 0.3147 - 0.3147e^{0.3147} = 0.5199$.

The value that we get is positive , so the root lies between 0.3147 and 1.

Taking $x_0 = 0.3147, x_1 = 1, f(x_0) = 0.5199, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.3147(2.1779) + 0.685335}{-2.17798 - (0.51987)} = \frac{1}{2.2384} = 0.44675 .$$

Now,

$$\cos 0.44675 - 0.44675e^{0.44675} = 0.2035 .$$

The value that we get is positive , so the root lies between 0.44675 and 1 .

Taking $x_0 = 0.44675, x_1 = 1, f(x_0) = 0.2035, f(x_1) = -2.17798$,

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.44675(-2.1779) - 1 \text{x} \, 0.2035}{-2.17798 - (0.2035)} = \frac{1}{2.0242} = 0.494020 .$$

Now,

$$\cos 0.494020 - 0.494020e^{0.494020} = 0.0708 .$$

The value that we get is positive , so the root lies between 0.494020 and 1 .

Taking $x_0 = 0.494020, x_1 = 1, f(x_0) = 0.0708, f(x_1) = -2.17798$

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.494020(-2.1779) - 1 \text{x} \, 0.0708}{-2.17798 - (0.0708)} = \frac{1}{1.9316} = 0.51771 \quad .$$

Now,

$$\cos 0.51771 - 0.51771 e^{0.51771} = 0.00124 \quad .$$

Taking , $x_0 = 0.51771, x_1 = 1, f(x_0) = 0.00124, f(x_1) = -2.17798$.

Using Formula ,

$$x_2 = \frac{x_0 y_1 - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.51771(-2.1779) - 1 \text{x} \, 0.00124}{-2.17798 - (0.0708)} = \frac{1}{1.9315699} = 0.5177136 \quad .$$

Now, $\cos 0.5177136 - 0.5177136 e^{0.5177136} = 0.00124$.

If we compare x_2 and x_0 , we find that both are same upto four decimal places .

Hence , the root of the equation correct upto four decimal places is 0.5177 .

Q6)c)1) Expand $\sin^4 \theta \cos^2 \theta$ in a series of multiples of Θ . (4M)

Ans : Let $x = \cos\theta + i\sin\theta$ $\quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

Also $x^n + \frac{1}{x^n} = 2\cos n\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Now consider ,

$$(2i\sin\theta)^4 (2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)^3 = \left(x^2 - \frac{1}{x^2}\right)^3 \left(x - \frac{1}{x}\right)^3$$

$$= \left(x^6 - 3x^2 + 3.\frac{1}{x^2} - \frac{1}{x^6}\right)\left(x - \frac{1}{x}\right) \quad .$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$(2i\sin\theta)^4(2\cos\theta)^3 = 2\cos7\theta - 2\cos5\theta - 6\cos3\theta + 6\cos\theta$$.

$$\sin^4\theta\cos^3\theta = \frac{\cos7\theta}{2^6} - \frac{\cos5\theta}{2^6} - \frac{3\cos3\theta}{2^6} + \frac{3\cos\theta}{2^6}$$.

Q6)c)2) If one root of $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$ is (1+i) ; find the other roots. (4M)

Ans : Since (1+ i) is a root of the given equation , then we know that (1-i) must be one of the remaining roots because complex roots always occur in conjugate pairs . Hence , (x-1-i) and (x-1+i) are the factors of the left hand side , i.e the left hand side is divisible by

{ (x-1)-i } .{ (x-1)+i } , i.e. by $(x-1)^2 - i^2$ = $x^2 - 2x + 2$.

Dividing the left hand side by $x^2 - 2x + 2$, we get $x^2 - 8x + 32$.

Solving the equation $x^2 - 8x + 32$, we get $x^2 = 4 \pm 4i$.

Hence , the remaining roots are $(1-i), (4+4i), (4-4i)$.
